Secure Fast Fourier Transform using Fully Homomorphic Encryption

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***Abstract*—Secure signal processing is becoming a de facto model for preserving privacy. We propose a model based on the Fully Homomorphic Encryption (FHE) technique to mitigate security breaches. Our framework provides a method to perform a Fast Fourier Transform (FFT) on a user-specified signal. Using encryption of individual binary values and FHE operations over addition and multiplication, we enable a user to perform the FFT in a fixed point fractional representation in binary. Our approach bounds the error of the implementation to enable user-selectable parameters based on the specific application. We verified our framework against test cases for one dimensional signals and images (two dimensional signals).**

***Keywords*—Image processing, computer security, Fast Fourier Transforms**

I. Introduction

Security breaches are severe situations that can cause significant problems when using cloud computing resources. This can occur because unencrypted data stored within these resources is vulnerable to security attacks, even if the cloud computing resource is trusted. Encrypting the data can mitigate such potential vulnerabilities. However, if the resources are being used to perform significant computations, then encrypting the data is not normally a possibility. Our focus is on solving the problem with the ability to process data while encrypted using Fully Homomorphic Encryption (FHE) [1]. FHE enables a user to encrypt their data and run a prescribed process against the encrypted data. In this paper we will focus on secure signal processing particularly, with performing the Fast Fourier Transform (FFT) using the FHE framework.

Use of FHE for secure signal processing is new technology and has potential for many more applications. Previous concepts focused on securing just the data. With FHE capabilities, it is possible to secure the data and process the data (i.e. perform a signal processing algorithm on it). Performing the signal processing algorithm while the data is encrypted enables the additional layer of security. This occurs because the system running the secure signal processing algorithm is not able to see the data and may not understand the process that is being used on the data [1]. Since signal processing is an often used technique in real world application, obfuscation of the data will be important and the method of FHE provides this. An additional note is the intermediate results of the algorithm are encrypted, so it is not possible for an attacker to gain any additional information.

Focusing on the overall problem, our user has a set of data that needs to be processed on a cloud computing resource. As illustrated in Fig. 1, the FHE process involves a user (client side) using a cloud computing resource (server side). The first step in the process is to generate keys for the encryption ((public, secret) key pair). With this key pair, the original signal can be encrypted (Step 2). The next step (3) in the process is to transport the data from the client to server; which is not a major focus of this paper, except the data should be transmitted via a secure channel to minimize exposure. Next, the encrypted process (Step 4) can occur (FFT in this paper). Here it is important to note that the processing on the server is actually developed by the user and transported to the server (this is not shown in the diagram). Similar to Step 3, Step 5 involves getting the data from the server back to the client (assumed secure connection). Finally (Step 6), the encrypted processed signal can be decrypted. The decryption results in the processed signal for the user. This process also leaves open the possibility of generating algorithms that are unknown to the client side. A complex algorithm that uses an FFT would need the building block of an encrypted FFT.

Considering the approach used by Shortell and Shokoufandeh [2], we improve it to perform an FFT in the encrypted space. Our approach focuses on using single binary digit encryption of fixed point values. This requires development of binary gate processors to take binary digit computations to full byte and word processing. Once this is done, it is possible to build an encrypted version of the FFT. This is analogous to building a CPU and having a computer program that performs the FFT via additions, subtractions, and multiplications. We verified our approach in one and two dimensional (image) cases of the FFT. Results of two dimensional are shown in Fig. 2.

The remainder of this paper will flow as follows. Section II presents the related work to encryption and performing secure signal processing. We discuss some background related to the FHE scheme used in Section III. Section IV provides the detailed discussion on implementation of FFT. In Sections V and VI we discuss the accuracy errors and our implementation

[1]Reverse engineering could be done to determine the algorithm. We assume the algorithm is not identified to the system (i.e. naming the process *Process10* vice *EncryptedFFT*)

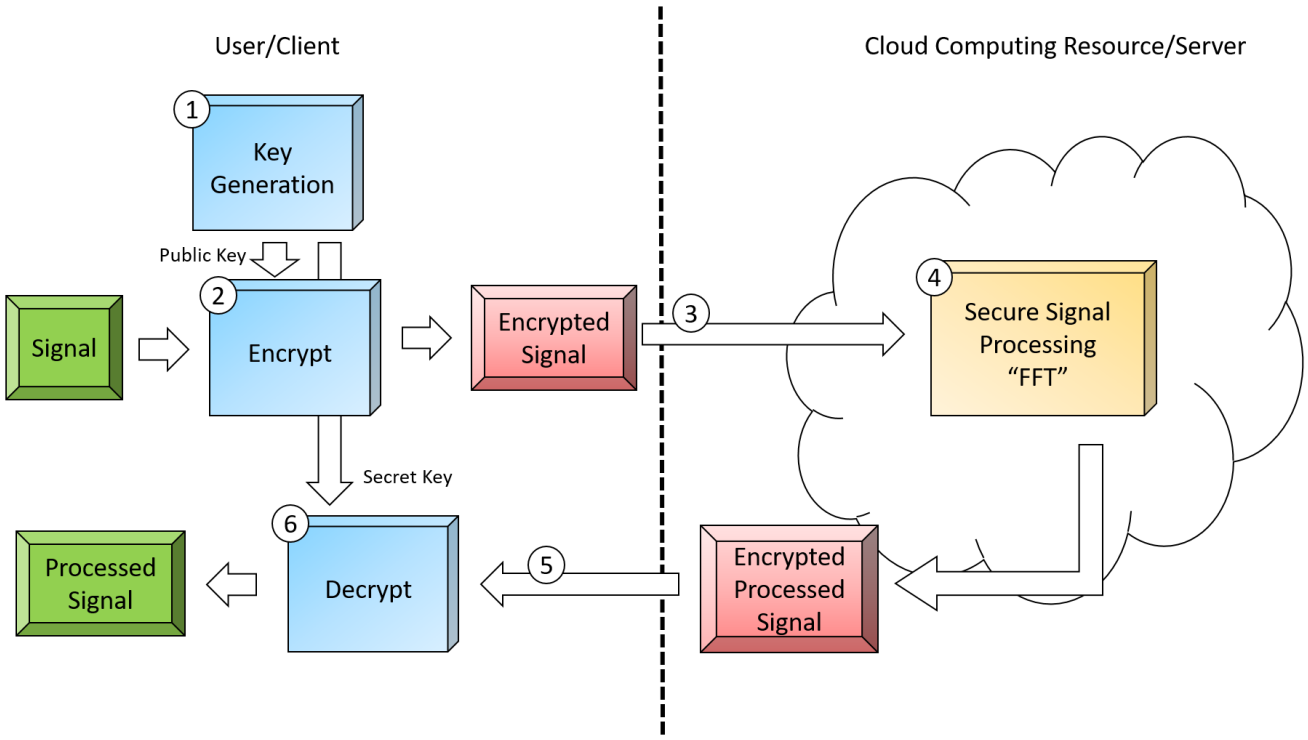

Fig. 1. FHE in cloud computing: Steps to perform FHE on a cloud computing resources from a data perspective.

evaluation results. We provide details on Time/Space Complexities of FFT over FHE in Section VII. Finally, we present our conclusions and future work in Section VIII.

II. Related Work

Research in secure signal processing has received increasing attention over the past decade. Troncoso-Pastoriza and Perez-Gonzalez [3] examined secure signal processing in the cloud very similar to the concept we use in our work. They focused on privacy issues that occur with cloud computing which is ideally what a Fully Homomorphic Encryption scheme can provide. Wang et. al. [4] also considered privacy issues with secure signal processing. Their focus was on biometrics and protecting authentication and privacy. This paper is similar in keeping confidentiality of private data in a cloud computing environment.

Other research in this field focuses on using the Paillier encryption scheme that provides homomorphic addition and constant multiplication. But the lack of ciphertext-ciphertext multiplication precludes this scheme from being fully homomorphic. Hsu, Lu, and Pei [5] used this scheme to perform the Scale Invariant Feature Transform (SIFT) in an encryption fashion. Bai et al. [6] also used the scheme for an encrypted SURF. There are a few other examples of using the Paillier scheme [7], [8]. While all of these examples can be performed in the encrypted domain, they fail to be a Fully Homomorphic Encryption. In contrast, our approach uses a Fully Homomorphic Encryption scheme.

It is also important to note some of the recent advances in Fully Homomorphic Encryption in the past few years. Gentry developed the original scheme in 2009 [1], [9]. The original scheme was designed for encrypting binary values and improvements over time continued to look at time and space complexity and the ability to encrypt more than just binary values [10], [11]. An improved scheme [12] developed a few years later is used in this paper as a FHE tool of choice for our solution

III. Notation and Background

In this paper we use small caps to identify individual binary gates. Z_q is used to represent an integer ring with a modulus of q. Letters are used for variables in error analysis equations. For Complex variables, we will use Re and Im to represent the real and imaginary values of a complex number.

To perform an encrypted FFT, we need the ability to encrypt and then process the ciphertexts. We use the Fully Homomorphic Encryption scheme defined and proposed by Gentry, Sahai, and Waters [12] that provides the basic capabilities including: key generation, encryption, decryption, and evaluation. Key generation provides a public/secret key pair that encrypts with the public key and decrypts with the secret key. Hardness of their scheme is based on learning with errors problem [13]; hence the secret key is a trapdoor in the public key to extract the original plaintext. Encryption and decryption are relatively straightforward conceptual processes. It is important to note that the ciphertexts are matrices that embed integers numbers, and the scheme can operate under the Z_q ring. Moving into evaluating ciphertexts, the scheme has four capabilities: addition, constant multiplication, two ciphertext multiplication, and a NAND gate. The first three operate over the Z_q ring, but last operation is only for binary values. We also rely on the fact the NAND gates can be combined into any other gate.

Fig. 2. Example of running FFT in encrypted domain for 2D images. The first row shows original images, the second row is the unencrypted FFT, third row is the FFT over FHE, and final row is the difference of the FFT results. As we can see in the last row our results showed that the 2D version of the FFT in the encrypted domain performs as well as the unencrypted version. The images have been scaled to aid in reviewing.

IV. FFT IN FHE

To implement FFT in the chosen FHE scheme, it is necessary to develop a structure that enables using the binary NAND gate to perform encrypted additions and constant multiplications. Since the individual ciphertexts are binary values, the additions and multiplications are going to need to be binary gates. Additionally, we need to perform calculations with fractional numbers given an arbitrary bit size. So the addition and multiplication processes in binary need to account for this.

Binary addition and multiplication are computed using different binary gates, mainly XOR and AND gates. NAND gates are extremely useful because all other binary gates can be calculated from them! This includes AND, OR, XOR, NOT, NOR, and XNOR. Having these gates available enables generation of more complex binary processes. Many of these gate computations are well known techniques.

Our next step is to use these binary gates and generate half and full adders. Having half and full adders will allow for performing an arbitrary bit size addition of two ciphertexts. Binary addition (and subtraction) is relatively straightforward. Full adders linked together starting from the lowest bit to the highest bit will generate the addition of two ciphertexts. Interestingly, subtraction can be performed by inverting the second binary values and using an initial carry bit of 1.

Next, we focus on constant multiplication and ciphertext-ciphertext multiplication. Binary multiplication causes the doubling of size for the binary values, which is an extremely important fact because of a two's complement implementation of binary numbers. Bit extension is necessary because otherwise negative numbers will not be calculated correctly. We used an implementation known as Wallace Trees for multiplication [14] as it provides a way to arbitrarily handle different binary sizes [2]. There is a potential security concern that may be introduced by constant multiplication. Because the constant is unencrypted, the binary values of the original constant can be partially traced into the Wallace tree computations. This can potentially provide an attacker knowledge about intermediary values. While this will not review the original ciphertext, it enables the attacker to identify values that the encrypted result cannot be. For example, if a few zeroes in the constant factor can identify a result bit to be zero, the space of potential values of the result has been reduced. Technically this is true of just multiplying an even constant factor because the result must be even.

At this point, we have methods to add, subtract, and multiply integers. The FFT algorithm will require computations involving floating point numbers. Our implementation uses fixed point numbers to emulate floating point numbers. By using a fixed point representation, we can use the integer based numbers to represent floating point numbers. This was similar to what was done in other work [2] but we improve this by

[2]For brevity, we refer the reader to [14] for implementation

using binary digits to handle cases that the straight integer implementation can't handle. Using a multiplier that is a factor of 2 (based on using binary values), floating point numbers can become fixed point numbers in an integer space. This approach introduces error, which we will discuss in the next section. For addition and subtraction, fixed point implementation is easily provided that the fractional bit size is consistent for both numbers (which we enforce). Multiplication is complicated because of the expansion in the bit size (doubling) and our solution is to extract a different set of the bits than just the lowest x (where x is the input bit size). By extracting the middle bits of the input size, an implicit division is occurring which provides a true implementation of fixed point multiplication. This also solves an earlier problem that caused long term expansion of fixed point multipliers during multiplication.

We have all the necessary building blocks required to implement FFT in the encrypted space. Step one of FFT is to perform a bit reversal, which is extremely easy since the order of the points is known and can easily be modified in position without revealing anything other than a bit reversal occurred. Step two requires calculating the W_n multipliers and running the butterfly computations over the signal for $\log N$ iterations ($N \log N$ driver). The multipliers are constants, so these can be input and used with constant multiplication. Constant multiplication results drive the additions for the signal points; yielding the two new points for the next iteration of the FFT. After N iterations of signal calculations and $\log N$ iterations of the outer loop, the process will be complete.

Theorem 1. *Fast Fourier Transform can be calculated in the encrypted domain via Fully Homomorphic Encryption.*

To prove Theorem 1, we need to show that the process does in fact perform the FFT in the encrypted domain. Following the process, we must prove that the encryption piece, the evaluation piece, and then the decryption piece work.

Lemma 1. *The FHE scheme properly encrypts values in the fixed fractional format to a vector of encrypted ciphertexts.*

Lemma 2. *The FHE scheme properly decrypts values in the fixed fractional format from a vector of ciphertexts.*

Lemma 3. *The FHE scheme properly evaluates the Fast Fourier Transform for binary vector ciphertexts of a vector of fixed fractional values.*

To prove Lemma 3, we need to show that the output of each binary vector is the same result as expected by the FFT algorithm. The key point here is that the butterfly computations of FFT provide the correct result. As we had shown earlier, addition, subtraction, and multiplication are needed to perform the butterfly computation. This becomes recursive as each operation is built from binary gates and finally at the NAND gate from the scheme itself. We start with individual lemmas for the individual gates and move up to three main operations. We assume the NAND gate is working as part of these lemmas. Theorem 3 of [12] proves the working of the FHE scheme basics (include Lemmas 1 and 2).

Lemma 4. *Given the* NAND *gate of the FHE scheme works, the FHE scheme properly performs a* NOT*,* AND*,* OR*, and* XOR *gates.* [3]

Proof: Previous research [15] has shown that a NAND gate can be used to generate all other logic gates including NOT, AND, OR, and XOR (which are important to this paper). Because the FHE scheme can compute a NAND gate, therefore we know the framework will correctly compute the remaining logic gates.

With Lemmas for the individual binary gates, we can create Lemmas for the half and full adders. Having the individual Lemmas will make the proofs significantly easier.

Lemma 5. *Given that a* XOR *and* AND *gate are available in the FHE scheme, the FHE scheme properly calculates a half adder.*

Proof: Since the half adder is a known working structure of binary arithmetic, our focus is to prove that it works in FHE. Lemma 4 proved FHE can correctly compute XOR and AND thus FHE can correct compute a half adder.

Lemma 6. *Given that an* OR *gate and a half adder are available in the FHE scheme, the FHE scheme properly calculates the full adder.*

Proof: Just as the half adder, a full adder is built from two half adders and an OR gate, this is a known correct concept. Lemma 4 proved the OR gate works in FHE. Lemma 5 proved a single half adder works in FHE. By construction, following the correct paths provides a full adder capability. It is now time to move on to addition and subtraction.

Lemma 7. *Given the binary gate and adder capabilities of the FHE scheme, the FHE scheme properly calculates arbitrary bit addition and subtraction of integers and fixed fractional format.*

Proof: Proving the scheme can correctly calculate addition and subtraction requires proving the binary result is correct. Since the algorithm is the same for both integers and the fixed fractional format, proving one proves both. We have the basic algorithm for adding and subtracting binary numbers and it is know to comprise a set of full adders. Lemma 6 proves the scheme can calculate a full adder correctly. Given this, we have proved that the addition and subtraction of ciphertexts occurs correctly in our FHE scheme.

Our next two lemmas are related to multiplication. Our first lemma will involve proving multiplication works for simple integer numbers including truncating the high order bits. The second lemma focuses on fixed fractional format and obtaining the correct resulting value. An interesting note at this point because of the lemma, is that binary division could actually be implemented but was not because it was not needed for FFT; however a single division is needed to support the fixed fractional format.

Lemma 8. *Given* AND *gates and half and full adders, the FHE scheme properly calculates integer multiplication via Wallace Trees.*

[3] We omit NOR and XNOR because they are not used for FFT.

Proof: Starting off, we know that the Wallace Trees correctly calculate integer multiplication of binary values. To prove our scheme calculates multiplication of two binary integers correctly, we need to show that we can correctly compute the Wallace Trees. Wallace Trees are built from a set of AND gates, followed by combinations of half and full adders. All three of these correctly compute binary values in our scheme based on our previous lemmas. The final importance of the proof is to show both positive and negative can be correctly calculated. This can be done easily by bit extending the binary values to twice the original size. Finally, we drop the higher order bits to keep the binary values size at their original size. Therefore, our scheme correctly calculates integer multiplication in the encrypted domain.

Lemma 9. *Given* AND *gates and half and full adders, the FHE scheme properly calculates fixed fractional format multiplication via Wallace Trees.*

Proof: We start with the fact that we can assume to have AND gate, half adder and a full adder that correctly compute their binary values in FHE. From Lemma 8, our scheme will properly calculate integer multiplication, which is part of our fixed fractional format. In a fixed number format, we need to divide the scaling factor out of the multiplied values, which is $2n$ of the previous size. Because we forced the multiplier to be a power of two, division is a simple bit shift. This is easy in our FHE scheme to drop the lower and higher order bits (extract the middle bits). Thus, the scheme calculates the fixed fractional format multiplication.

Using the above lemmas and their respective proofs, we can prove the evaluation lemma (3) which is the key butterfly computation of the FFT; without which the correctness of computations for FFT in encryption domain cannot be proven to be correct.

Proof: Remembering the two parts of the FFT, first is the bit reversal and then the butterfly computations. The bit reversal does not need to occur in the encrypted domain so this process works correctly as before. As the butterfly computations are processed in a loop and the loop processing is not encrypted, proving that the butterfly computations can be calculated in the encrypted domain will prove that the FFT can be calculated using FHE.

Simply, the butterfly computation is a pair of equations of complex numbers. Equivalently, there are six real number equations to calculate. These equations are a combination of addition, subtraction, and multiplication (via constant values). Lemmas 7 and 9 proved that the FHE scheme correctly calculates addition, subtraction, and multiplication of encrypted fixed fractional values. Because of this, we can calculate the butterfly computations and properly calculate the FFT.

Finally we can prove Theorem 1. *Proof:* To prove that FFT can be calculated via an FHE method, we focused on three pieces in the FHE scheme: Encryption, Evaluation, and Decryption. We have used three individual lemmas to prove that each one of these pieces is correctly computed in FHE (Lemmas 1, 3, and 2). Because we have proved the entire process works, we have proved the FFT can be computed in the encrypted domain using our FHE scheme.

V. Error Analysis

Having discussed the implementation of FFT using FHE, we turn our attention to estimating the error. Our only source of error is the conversion of floating point numbers to fixed point numbers. We note that the FHE scheme will use the term "error" as well. FHE error is not the same error as we discuss in this section. The FHE error is the primary security technique that enables the encryption and decryption, but the scheme itself accurately can encrypt and decrypt a single bit. Going back to the error introduced by the implementation, while initially this error might not be a major problem, over time the calculations will lose accuracy - the concept is no different than standard floating point operations in that over time accuracy can be lost. Immediately, we know that there will be dependencies on the number of points in the signal because of the dependency on the size with the number of iterations.

This error analysis is very important. This is the key enabler for making the framework a viable solution. Bounding the error enables the user to select parameters that will make the framework succeed. Without this analysis and thought, a user could easily run the algorithm and obtain inaccurate results without understanding why it failed.

We start with building up the bounds of the error introduced by the fixed point representation. There will some initial error caused by truncation when moving to fixed point representation. Following that, the main computation is the two point butterfly, which involves a complex point multiplication and an addition. We complete this section with a proof on the error bound for the entire FFT in FHE.

Lemma 10. *For a multiplication involving two complex points $(a + bi,\ c + di)$ with an error of Δ, $(0 \leq \Delta < 1)$ in each of the real and imaginary components, the error in the resultant is bounded by:*

$$\left(\Delta\left(a + c - b - d\right), \Delta\left(a + b + c + d\right)\right). \tag{1}$$

Proof: Our two initial points are:

$$(a + \Delta, b + \Delta) \tag{2}$$
$$(c + \Delta, d + \Delta) \tag{3}$$

Multiplying these together yields,

$$\begin{aligned}&((a + \Delta) \cdot (c + \Delta) - (b + \Delta) \cdot (d + \Delta),\\ &(b + \Delta) \cdot (c + \Delta) + (a + \Delta) \cdot (d + \Delta))\end{aligned} \tag{4}$$

As we expand the factors, we will drop terms of Δ^2 as these will not be the primary source of error in our final equations because $\Delta^2 < \Delta < 1$.

$$\begin{aligned}&(a \cdot d + \Delta \cdot (a + c) - (b \cdot d + \Delta \cdot (b + d)),\\ &b \cdot c + \Delta \cdot (b + c) + a \cdot d + \Delta \cdot (a + d))\end{aligned} \tag{5}$$

Continuing to combine terms,

$$\begin{aligned}&(a \cdot d - b \cdot d + \Delta \cdot (a + c - b - d),\\ &b \cdot c + a \cdot d + \Delta \cdot (b + c + a + d))\end{aligned} \tag{6}$$

This provides errors in the real and imaginary components as:

$$(\Delta \cdot (a + c - b - d), \\ \Delta \cdot (b + c + a + d)) \tag{7}$$

Next, we examine the error for a single butterfly computation. The original FFT equation is:

$$X_i = x_i + W_n * x_j. \tag{8}$$

Lemma 11. *For a single butterfly computation, the error is bounded by*

$$\Delta \cdot (Re(W_n) + Im(W_n) + Re(x_j) + Im(x_j) + 1) \tag{9}$$

for a single butterfly computation of a single point where Δ is the original error.

Proof: We start with the butterfly computation equation for a single point:

$$X_i = x_i + W_n \cdot x_j \tag{10}$$

Then adding Δ to each of the terms:

$$X_i = x_i + \Delta + (W_n + \Delta) \cdot (x_j + \Delta) \tag{11}$$

Lemma 10 provides the bound on the error for the right hand side of the equation. Then there is only addition of another Δ in both real and imaginary parts. This yields:

$$\Delta \cdot (\text{Re}(W_n) + \text{Re}(x_j) - \text{Im}(W_n) - \text{Im}(x_j) + 1) \\ \Delta \cdot (\text{Re}(W_n) + \text{Re}(x_j) + \text{Im}(W_n) + \text{Im}(x_j) + 1) \tag{12}$$

On the second half of the butterfly computations, the signs of the real and imaginary values are flipped (the one is always positive).

The previous two lemmas help us bound the overall error of the FFT by estimating the error accumulated over time. The most important item about the butterfly computations is that they reduce an $O(N^2)$ algorithm to a $O(N \log N)$ algorithm. This means the resulting value the FFT for a single point is a $O(N^2)$-based value that only uses $O(N \log N)$ computations. This helps bound the error from above: the summation of all signal points and the W_n values. But because we perform less computations, the value can be bounded lower.

Theorem 2. *For performing FFT in FHE, the error introduced by the processing is bounded by:*

$$\Delta \cdot \frac{N}{2} (\log N + X_b + 1), \tag{13}$$

where Δ is the original representation, W_S (used in proof) is a sum of all W_n that appear in the FFT for a given size N, and X_b is a bound on the size of the signal points.

Proof: Using Lemma 11, we know after a single butterfly for a value the error is:

$$\Delta \cdot (\text{Re}(W_n) + \text{Re}(x_j) - \text{Im}(W_n) - \text{Im}(x_j) + 1), \\ \Delta \cdot (\text{Re}(W_n) + \text{Re}(x_j) + \text{Im}(W_n) + \text{Im}(x_j) + 1). \tag{14}$$

After the second iteration of the outer loop of FFT, the W_n will be unchanged but the x_js will have additional values (and error). The key point here is what is being added to the error over time. It is actually the real and imaginary components of the signal points. This tells us at the second iteration, we have added up all W_n plus a number of terms from each x_j seen so far, and an equivalent number of ones from the x_i sides. After the $\log N$ iterations, each point will have the following error contributions:

$$\Delta \cdot \left(\sum_{W_n \in W} W_n + \sum_{\text{even}_j} x_j + \frac{N}{2} \right), \tag{15}$$

where we have used W to represent the set of W_n. Since each W_n is a known constant for a given N, we will call this sum bound as W_S. Additionally, we know that we can assume a bound on each x_j without loss of generality. Calling this value X_b and knowing there are $\frac{N}{2}$ of them, we can update the bound as:

$$\Delta \cdot \left(W_S + \frac{N}{2} (X_b + 1) \right) \tag{16}$$

An equivalent view on W_S parameter is that the W_n absolute values are less than one, which means W_S is bounded by the number of times any W_n enters an equation (i.e., $\frac{N \log N}{2}$) resulting in a total error of

$$\Delta \cdot \left(\frac{N}{2} (\log N + X_b + 1) \right). \tag{17}$$

This provides some different results than [2]. In particular, the error depends on the total number of points, which in turn means that the error will increase as the size of the signal increases. So when choosing a multiplier with the fixed fractional format, the signal size will matter. This is a good place to remember that an FHE scheme can evaluate ciphertexts indefinitely by refreshing ciphertexts (see [1] and [12] for details).

VI. Experimental Results

We implemented the encrypted FFT with FHE as a method to compare our theoretical analysis against a practical implementation. To verify our implementation works, we ran random signals against the FFT in FHE and compared the results to an unencrypted standard version FFT. Our main focus is to measure the error introduced by our implementation and verify Theorem 2 is valid. We used signal sizes of 8, 16, 32, 64, and 128 complex data signals. Additionally, we constrained the signal to be values in the range $[0, 1]$ similar to what many real world signals operate in. In the encrypted domain, we work in a 32-bit binary space and 16-bit fractional space. This means we limit our non-fractional integer size to 16-bits and we are using a multiplier of 65536. Having a multiplier of 65536 is useful since it will provide numerical accuracy up to $1/65536 \approx 1.5259e-5$. As we saw in the previous section, we will not be containing the entire set of values in this space because over time the processing will lose accuracy (particularly in multiplication).

It is important to look at measured error introduced as a whole for given signals. Our main focus is to make sure we can constrain the error. When comparing individual values between

the unencrypted and encrypted results, we can calculate the total error sum of all points ($2n$ from n complex points). We also calculate the mean error across the points along with the variance and standard deviation. Table I shows the results of the measured error from the random testing. One of the key aspects of the results is that the average error per point is slowly rising upwards above zero but is staying in the 10^{-5} range. This is expected as the error is dependent on the total number of points in the signal. Considering our accuracy of $1/65536$ and that the signal size is a multiplicative factor as well, experimentally the error is contained below the bounding from the previous section.

TABLE I. MEASURED ERROR IN ONE DIMENSIONAL TESTS

Complex Point Size	Total Error	Average Error	Variance	Standard Deviation
8 pt	0.0207	1.294e-5	8.208e-11	9.060e-06
16 pt	0.0709	2.216e-5	2.434e-10	1.560e-05
32 pt	0.258	4.199e-5	1.714e-09	4.140e-05
64 pt	1.030	8.383e-5	4.400e-09	6.633e-05
128 pt	4.437	1.81e-4	2.856e-08	1.69e-04

Finally, we tested our configuration using 16×16 images (10 total images; shown in Fig. 2). This provided a two dimensional test of the FFT over FHE in the multidimensional case. Table II shows the results of the ten images (mapped to the image order in Fig. 2). Overall the error was well contained across the images as the average total error was 0.0313. Average error across the images was $6.12e-5$ with a variance and standard deviation of $5.764e-9$ and $7.584e-5$. These values show our framework contains the error within bounds. Comparing these to the random generated signals for the one dimensional case identifies a few key points. Since the random signals had higher accuracy from the start, truncation of values occurred when going into the encrypted domain. The images initially had limited accuracy in comparison. Therefore when using the encrypted FFT, it is important to remember that the framework can represent certain number accurately and when the initial values do not contain much error the growth of error will not increase as fast.

TABLE II. MEASURED ERROR IN TWO DIMENSIONAL TESTS

Image	Total Error	Average Error	Variance	Standard Deviation
One	0.0308	6.020e-5	5.498e-9	7.415e-5
Two	0.0323	6.300e-5	6.803e-9	8.248e-5
Three	0.0294	5.751e-5	5.142e-9	7.171e-5
Four	0.0302	5.891e-5	5.982e-9	7.734e-5
Five	0.0325	6.343e-5	5.133e-9	7.164e-5
Six	0.0324	6.332e-5	5.480e-9	7.403e-5
Seven	0.0326	6.375e-5	5.227e-9	7.230e-5
Eight	0.0300	5.866e-5	6.122e-9	7.824e-5
Nine	0.0323	6.312e-5	6.011e-9	7.7529e-5
Ten	0.0308	6.017e-5	6.239e-9	7.899e-5

VII. TIME/SPACE COMPLEXITIES

FHE schemes are known to be computationally intensive. Encrypting a single value plaintext generates a two dimensional matrix in the ciphertext space. A NAND gate in the FHE scheme is an $O(N^3)$ process (matrix-matrix multiplication). This can be partially reduced by using parallelization techniques (GPU processing is real potential here). Focusing on FFT, which is an $O(M \log M)$ process, we need to understand the number of gates that can processed in total. $M \log M$ is the number of addition and multiplication gates being processed in total. A fixed point addition process is $36 * F$ NAND gates, where F is the size of fixed point size and 36 comes from the nand gates in the F sequential full adders. A fixed point multiplication process is $288F^2 \log F$ NAND gates. $4F$ AND gates for the first step and worst case assumed full adders for the $\log F$ process. This second process is for $2F$ full adders. It should be noted that this bound is worst case and can be significantly tightened because Wallace trees do not need to run full adders at each step. Combining these calculations yields an asymptotic running time of $O(M \log MF^2 \log FN^3)$ with constants removing, where we have used M to be the number of data points in the FFT, F to be the fixed point binary size, and N to be ciphertext size. Our implementation takes on the order of hours (˜2.5) to process a 16×16 image using parallel techniques available with FFT (processing multiple butterfly computations at once).

Coming back to the space complexity, we know a ciphertext text has N^2 space (matrix). We will set the size of the signal to be L, i.e. the total amount of data points in the signal whether single or multi-dimensional. Each data point has a set of F binary values. Multiplying these together yield: $O(LFN^2)$ space complexity. In both cases, these are extremely high complexities (especially compared to unencrypted processing). Our implementation showed that a 16×16 image while encrypted is about 87MB.

VIII. CONCLUSION

Having shown that we can perform the Fast Fourier Transform in the encrypted domain, we are now able to expand the capabilities of the FHE framework to target additional signal processing algorithms and other potential image processing algorithms like SIFT and SURF. There are other open issues with FHE. One major open item is that the FHE processing takes significant amount of time because of the matrix-matrix multiplication required for the underlying processing. Being able to improve computational performance of the FHE processing would contribute significantly to making FHE a viable solution in real world computing.

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