A Single-valued Pentagonal Neutrosophic Geometric Programming Approach to Optimize Decision Maker's Satisfaction Level

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***Abstract*—Achieving the desired level of satisfaction for a decision-maker in any decision-making scenario is considered a challenging endeavor because minor modifications in the process might lead to incorrect findings and inaccurate decisions. In order to maximize the decision-maker's satisfaction, this paper proposes a Single-valued Neutrosophic Geometric Programming model based on pentagonal fuzzy numbers. The decision-maker is typically assumed to be certain of the parameters, but in reality, this is not the case, hence the parameters are presented as neutrosophic fuzzy values. The decision-maker, with this strategy, is able to achieve varying levels of satisfaction and dissatisfaction for each constraint and even complete satisfaction for certain constraints. Here the decision maker aims to achieve the maximum level of satisfaction while maintaining the level of hesitation and minimizing dissatisfaction in order to retain an optimum solution. Furthermore, transforming the objective function into a constraint adds one more layer to the *N*-dimensional multi-parametrizes α, β and γ. The advantages of this multi-parametrized proposed method over the existing ones are proven using numerical examples.**

***Keywords*—Decision making; pentagonal neutrosophic numbers; single-valued neutrosophic geometric programming; multi-parametric programming**

I. Introduction

Mathematical optimization, an area of applied mathematics, is used to solve real-life issues by generating mathematical models to produce feasible outcomes. In today's world the significance of mathematical optimization and decision making can be explored in various fields [1-5]. Geometric Programming (GP) is a technique in the field of mathematical optimization and multi-objective decision making that is considered a significant optimization problem consisting of objective functions and constraints composed of monomials or posynomials that are designed to solve real-world engineering problems by generating feasible outcomes [6]. The basics of GP were initially introduced in a book by Duffin, Petersen and Zener [7], and afterward its improved and extended applications can be seen in various fields. Although many problems were solved by conventional GP, most of the time the problem contains uncertainties and is considered fuzzy rather than crisp. To deal with these Zadeh [8] introduced Fuzzy Sets (FS) which were later implemented in decision making by Bellman and Zadeh [9]. Tanaka and others [10] proposed fuzzy mathematical optimization by developing the notion of level sets. Later on, Zimmermann [11] introduced fuzzy linear programming using the concepts of fuzzy sets. Furthermore, the authors of [12] addressed series system models with the help of fuzzy parametric GP and achieved optimized system reliability and minimized cost. In the research of Samadi et al., [13] the authors presented an inventory model based on fuzzy GP for maximizing profit by implementing shortages.

Fuzzy decision-making models excel at addressing and optimizing decision-making problems, however evaluating membership values to our satisfaction is not always attainable due to a lack of readily available information. To overcome this issue, Atanassov [14] proposed Intuitionistic Fuzzy Set (IFS), which considers both membership and non-membership functions to effectively deal with vagueness rather than just the membership function as in fuzzy sets. Researchers then progressed with IFS in many real-life problems dealing with vague data, some of which are mentioned in the following literatures [15-18]. Smarandache later on introduced Neutrosophic sets (NS) [19] as the generalization of classical sets, FS and IFS which includes three independent membership functions representing the degrees of truth, indeterminacy and falsity for handling inconsistent, ambiguous and partial data more efficiently. However, the concept of NFS was established from a philosophical perspective, for which Wang et al., [20] introduced the notion of Single-valued Neutrosophic Sets (SvNS) to address practical, scientific and engineering challenges. Due to the limitations of the knowledge that humans acquire through experience or observation of the outside world, all the components indicated by SvNS are extremely appropriate for human consciousness. In contrast to the IFS, which cannot manage or represent indeterminacy and imprecise data, neutrosophic components are clearly the best fit in the representation of indeterminacy and inconsistent information. As a result, SvNS has quickly developed and is used in many different contexts [21-24]. With the advancement in research using SVNS many variations came into existence which includes triangular NS [25], trapezoidal NS [26] and recently pentagonal NS [27]. Das and Chakraborty initially applied pentagonal NS in solving linear programming problems by proposing a score function for converting the pentagonal NS data into crisp values. Further, Khalifa et al., [28] applied pentagonal neutrosophic based linear programming for optimizing stock portfolio investment. The authors of [29] worked on maximizing profit using EOQ models using pentagonal neutrosophic demands.

A. Gaps in Existing Research and Contribution

The aspect of decision-making can be seen in many domains including geometric programming where several researchers contributed their works by developing and presenting diverse techniques to solve complex decision making problems and finding optimum solutions.

In the previous scenarios, the expert person was introduced with the simple single vector α whose value would be constant for each constraint which bound the expert to provide the same level of satisfaction to the decision maker for every constraint [30]. Thereby the expert is in a dilemmatic situation where he needs to satisfy the decision-maker but without compromising the optimal solution resulting in following extra steps for the sake of optimality. This scenario is tackled in our work where the optimal solution is achieved while providing satisfaction to the decision maker.

A multi-parametric programming approach was introduced in [31] for reaching the optimal solution in Linear Programming Problems (LPP). They proposed a method comprising a vector that would help the decision maker to attain a better satisfactory level for LPP. Though, the authors here employed an n-dimensional vector to obtain an optimal solution but did not reach the highest level of satisfaction of the decision maker in Geometric Programming Problems (GPP) since they did not consider fuzzy numbers for their work. So to overcome this, [36] proposed a multi-parametric vector α based on fuzzy numbers to solve the geometric programming problems to deal with the vagueness present in the decision making scenario. Here they were able to reach even the complete satisfaction for the decision maker in certain constraints. As a result, the expert is able to satisfy the decision maker for each constraint while maintaining the optimal value in the fuzzy GPP. Unfortunately, Fuzzy numbers cannot deal with indeterminacy, which is why Neutrosophic numbers are used in our proposed model.

However, the term "decision making" doesn’t always indicate “identifying the best output from any programming problem”. Instead, the decision-maker aims to achieve the intended level of satisfaction, which may or may not be the same as maximizing or minimizing the objective function. As a result, the constraint has a different effect than in the standard version. The previous attempts in multilevel decision making are mostly focused on identifying the ideal circumstances and solution algorithms to tackle linear, nonlinear, and discrete elementary problems, with only one decision allocated to each decision level for optimizing the final distinct objective. Therefore, this study focuses on maximizing satisfaction and minimizing dissatisfaction levels of the decision-maker while keeping in check the hesitation levels by incorporating multi-parametric vectors α , β and γ to Single-valued Neutrosophic Geometric Programming (SvNGP). Furthermore, the pentagonal neutrosophic numbers are subjected to a score function in order to establish a link between coefficients and exponents and obtain the crisp values. In this regard, the primary contributions of the proposed decision making model are as follows:

1) There is not much effort put towards improving decision-maker’s satisfaction while taking a decision, particularly in neutrosophic environments. Thus the proposed approach aids decision-maker to take firm and confident decisions.

2) The use of pentagonal neutrosophic numbers aids in coping with imprecision and results in achieving robust decisions.

3) The objective function has been transformed into a constraint. As a result, the solution begins with the initial optimal point.

4) The addition of a new constraint to the SvPNGP problem adds a new dimension as well as a new restriction on the feasible solution space. As a result, the proposed multi-parametric α , β and γ vectors comprise $(N + 1)$ dimensions.

5) This technique allows the decision-maker to place his desires on each constraint individually, offering him more flexibility.

6) Inclusion of tolerance value aids in achieving precise results while using SvPNGP.

7) It can deal with uncertainties, hesitancies and inconsistent data more efficiently.

8) The decision-maker can manage the satisfaction, hesitation and dissatisfaction degrees resulting in reaching his/her maximum desire.

9) The proposed approach is applicable to real-life programming problems.

The rest of the paper is described as: Section II presents the preliminary definitions and theorems. To generate crisp values from Single-valued pentagonal neutrosophic numbers, a score function is taken into consideration which is described in Section III. In Section IV, multi-parametric vectors $\alpha, \beta, \gamma(0,1]^{N+1}$ are introduced to evaluate the optimum solution and values for the SvPNGP problem. Certain membership, non-membership and indeterminacy functions are modeled specifically for the programming problem and related theorems are also studied. In Section V, concepts of feasibility and efficiency using multi-parametric programming is described. Section VI discusses the two-phase strategy as well as the proposed algorithm to solve the SvPNGP problem. The efficacy of the method given is assessed using a numerical example problem in Section VII and the findings are examined with other methods. Also, the advantages of our method compared to the other methods are also pointed out. At last, in Section VIII, the concluding remarks are given.

II. Preliminaries

In this section, several definitions and theorems are discussed that could be useful for analysis.

Definition 1. [8] A set of ordered pairs $\tilde{S}$ is said to be a fuzzy set if:

$$\tilde{S} = \{(x, \mu_{\tilde{S}}(x)) | x \in X\}$$

where X is a non-empty set and the function $\mu_{\tilde{S}} : X \rightarrow [0,1]$ denotes the membership function of $\tilde{S}$.

Definition 2. [14] Let $A' \in X$ be an intuitionistic fuzzy set (IFS) which is defined as a triplet in the form:

$$A' = \{(x, \mu_{A'}(x), \vartheta_{A'}(x)) | x \in X\}$$

where $\mu_{A'}(x), \vartheta_{A'}(x): X \to [0,1]$ such that $0 \le \mu_{A'}(x) + \vartheta_{A'}(x) \le 1$. The function $\mu_{A'}(x)$ represents the degree of membership and $\vartheta_{A'}(x)$ represents the degree of non-membership for every $x \in X$. Moreover, a hesitation margin or intuitionistic fuzzy index $\pi_{A'}$ can be defined as $\pi_{A'} = 1 - \mu_{A'}(x) - \vartheta_{A'}(x)$ for all $x \in A'$ which indicates the degree of belongingness of x in A'.

Definition 3. [19] A neutrosophic set $\tilde{P} \in X$ is defined by:

$$\tilde{P} = \{(x, \mu_{\tilde{P}}(x), \sigma_{\tilde{P}}(x), \vartheta_{\tilde{P}}(x)) | x \in X\}$$

where X is an universal set and $\mu_{\tilde{P}}(x), \sigma_{\tilde{P}}(x), \vartheta_{\tilde{P}}(x)$ represents three functions namely membership, indeterminacy and non-membership respectively. Their bounds are defined as:

$$\mu_{\tilde{P}}(x), \sigma_{\tilde{P}}(x), \vartheta_{\tilde{P}}(x): X \to]0^-, 1^+[$$

$$\text{such that } 0^- \le \mu_{\tilde{P}}(x) + \sigma_{\tilde{P}}(x) + \vartheta_{\tilde{P}}(x) \le 3^+$$

Definition 4. [20] A single valued neutrosophic set $\tilde{P} \in X$ is defined by:

$$\tilde{P} = \{(x, \mu_{\tilde{P}}(x), \sigma_{\tilde{P}}(x), \vartheta_{\tilde{P}}(x)) | x \in X\}$$

where X is an universal set and $\mu_{\tilde{P}}(x), \sigma_{\tilde{P}}(x), \vartheta_{\tilde{P}}(x)$ represents three functions namely membership, indeterminacy and non-membership respectively. Their bounds are defined as:

$$\mu_{\tilde{P}}(x), \sigma_{\tilde{P}}(x), \vartheta_{\tilde{P}}(x): X \to [0,1]$$

$$\text{such that } 0 \le \mu_{\tilde{P}}(x) + \sigma_{\tilde{P}}(x) + \vartheta_{\tilde{P}}(x) \le 3$$

Definition 5.[32] Let a single-valued pentagonal neutrosophic number (SvPNN) be defined as $\tilde{r} = [(r_1, r_2, r_3, r_4, r_5); \mu_{\tilde{r}}], [(s_1, s_2, s_3, s_4, s_5); \sigma_{\tilde{r}}], [(t_1, t_2, t_3, t_4, t_5); \vartheta_{\tilde{r}}]$, such that $\tilde{r} \in \mathbb{R}$, where $\mathbb{R}$ is a set of real numbers and $\mu_{\tilde{r}}, \sigma_{\tilde{r}}, \vartheta_{\tilde{r}} \in [0,1]$. Then the membership function $\mu_{\tilde{r}}(x): \mathbb{R} \to [0, \mu]$, indeterminacy function $\sigma_{\tilde{r}}(x): \mathbb{R} \to [\sigma, 1]$ and non-membership function $\vartheta_{\tilde{r}}(x): \mathbb{R} \to [\vartheta, 1]$ of $\bar{r}$ is given by:

$$\mu_{\tilde{r}}(x) = \begin{cases} \mu_{\widetilde{rl1}}(x) & r_1 \le x \le r_2 \\ \mu_{\widetilde{rl2}}(x) & r_2 \le x \le r_3 \\ \mu & r_3 \\ \mu_{\widetilde{ru1}}(x) & r_3 \le x \le r_4 \\ \mu_{\widetilde{ru2}}(x) & r_4 \le x \le r_5 \\ 0 & otherwise \end{cases}$$

$$\sigma_{\tilde{r}}(x) = \begin{cases} \sigma_{\widetilde{rl1}}(x) & s_1 \le x \le s_2 \\ \sigma_{\widetilde{rl2}}(x) & s_2 \le x \le s_3 \\ \sigma & s_3 \\ \sigma_{\widetilde{ru1}}(x) & s_3 \le x \le s_4 \\ \sigma_{\widetilde{ru2}}(x) & s_4 \le x \le s_5 \\ 0 & otherwise \end{cases}$$

$$\vartheta_{\tilde{r}}(x) = \begin{cases} \vartheta_{\widetilde{rl1}}(x) & t_1 \le x \le t_2 \\ \vartheta_{\widetilde{rl2}}(x) & t_2 \le x \le t_3 \\ \vartheta & t_3 \\ \vartheta_{\widetilde{ru1}}(x) & t_3 \le x \le t_4 \\ \vartheta_{\widetilde{ru2}}(x) & t_4 \le x \le t_5 \\ 0 & otherwise \end{cases}$$

A graphical illustration of linear pentagonal neutrosophic number can be seen in Fig. 1. Here the three lines viz., black, red and blue represents the membership, non-membership and indeterminacy functions respectively. Here, the variable τ is represented by the notation $0 \le \tau \le 1$, where the pentagonal number will become a triangular neutrosophic number if τ = 0 or 1.

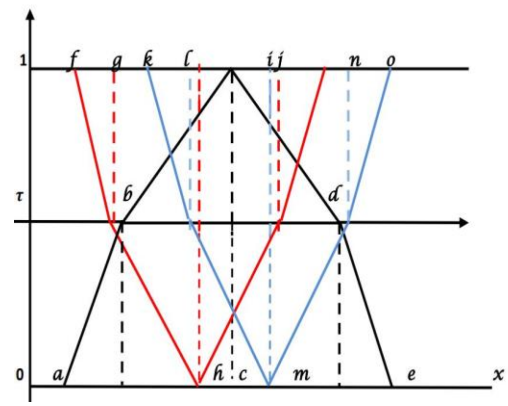

Fig. 1. Pictorial form of linear pentagonal neutrosophic number [32].

III. Crispification of SvPNN

To transform neutrosophic numbers into crisp values, score and accuracy functions are required. We adopted the notion of score and accuracy function from [27] for a SvPNN $\tilde{f}_{PN} = (f_1, f_2, f_3, f_4, f_5; \mu, \sigma, \vartheta)$ which is defined a as follows:

1) *Score function:* The score function for $\tilde{f}_{PN}$ is scaled as $\widetilde{Sc}_{PN} = \frac{1}{15}(f_1 + f_2 + f_3 + f_4 + f_5) \text{x} \{2 + \mu - \sigma - \vartheta\}$
2) *Accuracy function:* The accuracy function is given as $\widetilde{Ac}_{PN} = \frac{1}{15}(f_1 + f_2 + f_3 + f_4 + f_5) \text{x} \{2 + \mu - \sigma\}$

IV. Single-valued Neutrosophic Geometric Programming

Definition 6.[33] The standard form of Posynomial Geometric Programming (PGP) of X is given by:

$$\max \quad \sum_{s=1}^{q_o} v_{0s} \prod_{j=1}^{M} X_j^{\gamma_{0sj}}, \tag{1}$$

$$\text{s.t.} \quad \sum_{s=1}^{q_i} v_{is} \prod_{j=1}^{M} X_j^{\gamma_{isj}} \le h_i, \quad i = 1, \ldots, N$$

$$X_j > 0$$

where the M-dimensional variable $X = (X_1, \dots, X_M)^T > 0$, coefficients $v_{is} > 0$ and exponents $\gamma_{isj} > 0$, are arbitrary real numbers.

The problem

$$\widetilde{\max} \quad \tilde{g}_0(X), \tag{2}$$
$$\text{s.t.} \quad \tilde{g}_i(X) \lesssim 1, \quad i = 1, \dots, N$$
$$X > 0$$

is called an Single-valued Neutrosophic PGP (SvNPGP) problem, where $g_i(X) = \sum_{s=1}^{q_i} v_{is}(X), i = 0, \dots, N$, is a posynomial function on X, where the monomial function v_{is} of X is defined as [34]:

$$v_{is} = \begin{cases} v_{is} \prod_{j=1}^{M} X_j^{\gamma_{isj}}, & s = 1, \dots, \varrho_i, i = 1, \dots, N' \\ v_{is} \prod_{j=1}^{M} X_j^{-\gamma_{isj}}, & s = 1, \dots, \varrho_i, i = N' + 1, \dots, N \end{cases}$$

and $\mathcal{Z}_0 \lesssim g_0(X) \to \widetilde{max}\, g_0(X)$ represents the maximum goal of the objective function $g_0(X)$ where $\mathcal{Z}_0$ is considered as the lower bound. $\mathcal{Z}_0$ is the expectation value of $g_0(X)$ and " $\lesssim$ "symbolizes fuzzy version of " $\leq$ " which basically means “less than or equal to”. Therefore (2) can be changed into Single-valued neutrosophic reversed PGP problem:

$$\widetilde{\max} \quad \widetilde{g_0}(X) \gtrsim \mathcal{Z}_0, \tag{3}$$
$$\text{s.t} \quad \widetilde{g_i}(X) \lesssim 1, \qquad i = 1, \dots, N'$$
$$\widetilde{g_i}(X) \gtrsim 1, \qquad i = N' + 1, \dots, N$$
$$X > 0$$

Definition 7. A monomial function of PGP can be defined as fully SvNPGP form as:

$$\widetilde{\max} \quad \tilde{v}_0 \prod_{j=1}^{M} \tilde{X}_j^{\tilde{\gamma}_{0j}}, \tag{4}$$
$$\text{s.t} \quad \tilde{v}_i \prod_{j=1}^{M} \tilde{X}_j^{\tilde{\gamma}_{ij}} \leq \tilde{h}_i, \qquad i = 1, \dots, N$$
$$\tilde{X} > \tilde{0}$$

where all the coefficients $\tilde{v}_i > \tilde{0}$ for $i = 1, \dots, N$, variables $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_M)^T$, exponents $\tilde{\gamma}_{ij}$ and real numbers $\tilde{h}_i > \tilde{0}$ are Single-valued Neutrosophic numbers.

Theorem 1 .[35] Let $g_i(X)$ be a convex function for any i, then the resulting geometric programming problem is an Single-valued Neutrosophic convex problem

$$g_0(X) \lesssim g_0 \tag{5}$$
$$g_i(X) \lesssim 1, \qquad i = 1, \dots, N$$

Theorem 2. Any SvNPGP can be turned into a Single-valued neutrosophic convex programming problem, as specified in (2).

Proof. Let $T_j = \log(X_j)$, where $T_j = (\langle T_{j\mu}, T_{j\sigma}, T_{j\vartheta} \rangle)$, so $X_j = e^{T_j}$ for $1 \leq j \leq M$. Then

$$\sum_{s=1}^{q_i} v_{is} \prod_{j=1}^{M} X_j^{\gamma_{isj}} = \sum_{s=1}^{q_i} v_{is}\, e^{\sum_{j=1}^{m} T_j \gamma_{isj}} = g_i(X), \quad i = 0, \dots, N \tag{6}$$

Thereby, problem (2) can be turned into (5). So, by applying Theorem 1, we are able to prove it.

Theorem 3. Any Single-valued Neutrosophic monomial PGP problem (4) can be converted into a Single-valued Neutrosophic linear programming problem as follows:

$$\max \quad ln\tilde{v}_0 + \sum_{j=1}^{M} \gamma_{0j} \tilde{T}_j, \tag{7}$$
$$\text{s.t} \quad ln\tilde{v}_i + \sum_{j=1}^{M} \gamma_{ij} \tilde{T}_j \leq ln\tilde{\mathrm{Q}}_j, \qquad i = 1, \dots, N$$
$$\tilde{X}_j > \tilde{0}, \qquad j = 1, \dots, M$$

Proof. By using “*ln*” function on (4), we can say that:

$$\max \quad ln\tilde{v}_0 + \sum_{j=1}^{M} \gamma_{0j} ln\tilde{X}_j, \tag{8}$$
$$\text{s.t} \quad ln\tilde{v}_i + \sum_{j=1}^{M} \gamma_{ij} ln\tilde{X}_j \leq ln\tilde{\mathrm{Q}}_j, \qquad i = 1, \dots, N$$
$$\tilde{X}_j > \tilde{0}, \qquad j = 1, \dots, M$$

Now, when we put $ln\tilde{X}_j = \tilde{T}_j$ in (8), a convex program is obtained as follows:

$$\max \quad ln\tilde{v}_0 + \sum_{j=1}^{M} \gamma_{0j} \tilde{T}_j, \tag{9}$$
$$\text{s.t} \quad ln\tilde{v}_i + \sum_{j=1}^{M} \gamma_{ij} \tilde{T}_j \leq ln\tilde{\mathrm{Q}}_j, \qquad i = 1, \dots, N$$
$$\tilde{X}_j > \tilde{0}, \qquad j = 1, \dots, M$$

Thus, from theorem 2, we can say that the above problem is a convex programming problem and has a similar Single-valued Neutrosophic optimal solution as the problem (4).

V. Feasibility and Efficiency Concepts through Multi-parametric Programming

The notion of a multi-parametric vector (α,β,γ) is introduced in this section which is useful to evaluate the level of confidence derived from the feasibility and efficacy of the optimum solution. Inclusion of tolerance value to the programming problem as a novel membership function imposes limitation as a prerequisite that can play a significant part in obtaining a suitable solution. Furthermore, the decision maker's satisfaction will be closer to the feasible solution. Considering (2) and assuming that $\tilde{A}_i$ represents every X of neutrosophic constraints related to the neutrosophic inequality constraint $g_i(X) \lesssim 1, (i = 1,\dots,N)$, the membership function $\mu_{\tilde{A}_i}(X)$, indeterminacy function $\sigma_{\tilde{A}_i}(X)$ and non-membership function $\vartheta_{\tilde{A}_i}(X)$ are given by:

$$\mu_{\tilde{A}_i}(X) = \begin{cases} 1, & g_i(X) \le 1 \\ 1 - \dfrac{d_i}{t_i}, & g_i(X) = 1 - d_i \; (d_i = 0,\dots,t_i) \\ 0, & g_i(X) \ge 1 - t_i \end{cases}$$

$$\sigma_{\tilde{A}_i}(X) = \begin{cases} 0, & g_i(X) \le 1 \\ \dfrac{1 - d_i}{t_i}, & g_i(X) = 1 - d_i \; (d_i = 0,\dots,t_i) \\ 1, & g_i(X) \ge 1 - t_i \end{cases}$$

$$v_{\tilde{A}_i}(X) = \begin{cases} 0, & g_i(X) \le 1 \\ \dfrac{1 - d_i}{t_i}, & g_i(X) = 1 - d_i \; (d_i = 0,\dots,t_i) \\ 1, & g_i(X) \ge 1 - t_i \end{cases}$$

where $t_i \in \mathbb{R}^+$ represents the maximum tolerance value which is determined by the decision-maker. The decision-maker assigns a tolerance value which can complicate the SvNGP problem. So, selecting a tolerance value throughout the decision making process, aiming to please the decision-maker, and then enhancing his satisfaction level, ultimately boosts efficiency.

By observing problem (3), multi-parametric vectors α,β and γ are presented where $\alpha = (\alpha_0,\dots,\alpha_N) \in (0,1]^{N+1}$, $\beta = (\beta_0,\dots,\beta_N) \in (0,1]^{N+1}$ and $(\gamma_0,\dots,\gamma_N) \in (0,1]^{N+1}$ represents the confidence level for the membership, non-membership and indeterminate values respectively of the programming problem. Here α_0,β_0 and γ_0 represents the satisfaction, dissatisfaction and hesitation degrees respectively, for the objective function which then will be converted into a constraint imposing a limitation to the feasible solution resulting a precise optimal solution whereas α_N, β_N and γ_N for i = 1,…, N represents the satisfaction, dissatisfaction and hesitation degrees for each constraint. Thus, a new membership, indeterminacy and non-membership function is created solely for the objective function after it is converted to a constraint which is defined as:

$$\mu(g_0(X,\alpha,\beta,\gamma)) = \frac{\mathcal{Z}_0 - g_0(X,\alpha,\beta,\gamma)}{t_0}, \qquad g_0(X,\alpha,\beta,\gamma) \ge \mathcal{Z}_0 - t_0 \tag{10}$$

$$\sigma(g_0(X,\alpha,\beta,\gamma)) = \frac{g_0(X,\alpha,\beta,\gamma) - \mathcal{Z}_0 + t_0}{t_0}, \qquad g_0(X,\alpha,\beta,\gamma) \ge \mathcal{Z}_0 - t_0$$

$$v(g_0(X,\alpha,\beta,\gamma)) = \frac{g_0(X,\alpha,\beta,\gamma) - \mathcal{Z}_0 + t_0}{t_0}, \qquad g_0(X,\alpha,\beta,\gamma) \ge \mathcal{Z}_0 - t_0$$

and

$$\mu(g_i(X,\alpha,\beta,\gamma)) = \begin{cases} 1, & g_i(X,\alpha,\beta,\gamma) \le b_i \\ 1 - \dfrac{g_i(X,\alpha,\beta,\gamma) - b_i}{t_i}, & b_i \le g_i(X,\alpha,\beta,\gamma) \le b_i + t_i, \; i = 1,\dots,N' \\ 0, & g_i(X,\alpha,\beta,\gamma) \ge b_i + t_i, \; i = N'+1,\dots,N \end{cases}$$

$$\sigma(g_i(X,\alpha,\beta,\gamma)) = \begin{cases} 0, & g_i(X,\alpha,\beta,\gamma) \le b_i \\ \dfrac{g_i(X,\alpha,\beta,\gamma) - b_i}{t_i}, & b_i \le g_i(X,\alpha,\beta,\gamma) \le b_i + t_i, \; i = 1,\dots,N' \\ 1, & g_i(X,\alpha,\beta,\gamma) \ge b_i + t_i, \; i = N'+1,\dots,N \end{cases}$$

$$v(g_i(X,\alpha,\beta,\gamma)) = \begin{cases} 0, & g_i(X,\alpha,\beta,\gamma) \le b_i \\ \dfrac{g_i(X,\alpha,\beta,\gamma) - b_i}{t_i}, & b_i \le g_i(X,\alpha,\beta,\gamma) \le b_i + t_i, \; i = 1,\dots,N' \\ 1, & g_i(X,\alpha,\beta,\gamma) \ge b_i + t_i, \; i = N'+1,\dots,N \end{cases}$$

Thus an underlying framework is presented to discover the optimum solution in terms of the satisfaction, dissatisfaction and indeterminacy degrees of the decision-maker with the maximum tolerance in (3):

$$\begin{array}{lll} \max & g_0(X), & \\ \text{s.t} & g_0(X) \ge \mathcal{Z}_0 - \alpha_0 t_0, & \\ & g_0(X) \ge \mathcal{Z}_0 + (1-\gamma_0)t_0, & \\ & g_0(X) \ge \mathcal{Z}_0 + (1-\beta_0)t_0, & \\ & g_i(X) \le b_i + (1-\alpha_i)t_i, & i = 1,\dots,N' \\ & g_i(X) \le b_i + \gamma_i t_i, & i = 1,\dots,N' \\ & g_i(X) \le b_i + \beta_i t_i, & i = 1,\dots,N' \\ & g_i(X) \ge b_i + (1-\alpha_i)t_i, & i = N'+1,\dots,N \\ & g_i(X) \ge b_i + \gamma_i t_i, & i = N'+1,\dots,N \\ & g_i(X) \ge b_i + \beta_i t_i, & i = N'+1,\dots,N \\ & X > 0, \alpha_i,\beta_i,\gamma_i \in (0,1], & i = 0,\dots,N \\ & \alpha \ge \gamma, \alpha \ge \beta, \alpha+\beta+\gamma \le 3 & \end{array} \tag{11}$$

Definition 8. Let $X^* = (X_1{}^*,\dots,X_M{}^*)^T \in \mathbb{R}^M$ be an M-dimensional vector where $\alpha,\beta,\gamma \in [0,1]$ and $\alpha+\beta+\gamma \le 3$, defined as:

$$X_{\alpha,\beta,\gamma} = \left\{ X \in \mathbb{R}^M \;\middle|\; X \ge 0, \; \begin{array}{l} \mu_i(g_i(X)) \ge \alpha_i \\ \vartheta_i(g_i(X)) \le \beta_i, i = 0,\dots,N \\ \sigma_i(g_i(X)) \le \gamma_i \end{array} \right\} \tag{12}$$

in which a vector $X \in X_{\alpha,\beta,\gamma}$ will be an α,β,γ-feasible solution for (2) where α is the minimal acceptance degree, β and σ denotes the maximum rejection and hesitation degree respectively.

Theorem 4. Let $\alpha = (\alpha_0,\dots,\alpha_N) \in (0,1]^{N+1}, \beta = (\beta_0,\dots,\beta_N) \in (0,1]^{N+1}$ and $\gamma = (\gamma_0,\dots,\gamma_N) \in (0,1]^{N+1}$, and for $X_j^* \ge 0, j = 1,\dots N$, $X^* = (X_1{}^*,\dots,X_M{}^*)^T \in \mathbb{R}^M$ is an M-dimensional vector and an α,β,γ-feasible solution for (2). So X^* is an α,β,γ-efficient optimal solution iff it satisfies the following constraints -

$$\begin{array}{ll} \max & g_0(X) \\ \text{s.t} & g_0(X) \ge \mathcal{Z}_0 - \alpha_0 t_0, \\ & g_0(X) \ge \mathcal{Z}_0 + (1-\gamma_0)t_0, \end{array} \tag{13}$$

$g_0(X) \geq \mathcal{Z}_0 + (1-\beta_0)t_0,$

$g_i(X) \leq 1 + (1-\alpha_i)t_i, \quad i = 1, \ldots, N'$

$g_i(X) \leq 1 + \gamma_i t_i, \quad i = 1, \ldots, N'$

$g_i(X) \leq 1 + \beta_i t_i, \quad i = 1, \ldots, N'$

$g_i(X) \geq 1 + (1-\alpha_i)t_i, \quad i = N'+1, \ldots, N$

$g_i(X) \geq 1 + \gamma_i t_i, \quad i = N'+1, \ldots, N$

$g_i(X) \geq 1 + \beta_i t_i, \quad i = N'+1, \ldots, N$

$X > 0, \alpha_i, \beta_i, \gamma_i \in (0,1], \quad i = 0, \ldots, N$

$\alpha \geq \gamma, \alpha \geq \beta, \alpha + \beta + \gamma \leq 3$

where t_i denotes the maximum tolerance.

Proof. Let us consider that $= (\alpha_0, \ldots, \alpha_N) \in (0,1]^{N+1}, \beta = (\beta_0, \ldots, \beta_N) \in (0,1]^{N+1}$ and $\gamma = (\gamma_0, \ldots, \gamma_N) \in (0,1]^{N+1}$, and for $X_j^* \geq 0, j = 1, \ldots N$, $X^* = (X_1^*, \ldots, X_M^*)^T \in \mathbb{R}^M$ is an α, β, γ-feasible solution for (2). From definition 8 and problem (9), we have $\mu_i\big(g_i(\mathcal{X})\big) \geq \alpha_i, v_i\big(g_i(\mathcal{X})\big) \leq \beta_i$ and $\sigma_i\big(g_i(\mathcal{X})\big) \leq \gamma_i$, therefore X^* is a feasible solution. However, as $X^* \in \mathbb{R}^M$ is an α, β, γ-efficient solution, no other $X^{*'} \in \mathcal{X}_{\alpha,\beta,\gamma}$ will satisfy $g_0\big(X^{*'}\big) > g_0(X^*)$. Thus, it means $X^{*'}$ is an optimal solution. Moreover, if we consider $X^{*'}$ be an optimal solution for (12) and apparently, $X^{*'}$ is an α, β, γ-feasible solution, it means $X^{*'}$ is an α, β, γ-efficient solution.

Now, let us assume the optimal solution for problem (12) be $(\mathcal{X}^*, \mathcal{Z}_0 = g_0(\mathcal{X}^*))$ in Theorem 3. It is only necessary to solve the programming problem below:

$$\max \sum_{i=1}^{N} \alpha - \beta - \gamma \tag{14}$$

s.t

$g_0(X) \geq \mathcal{Z}_0 - \alpha_0 t_0,$

$g_0(X) \geq \mathcal{Z}_0 + (1-\gamma_0)t_0,$

$g_0(X) \geq \mathcal{Z}_0 + (1-\beta_0)t_0,$

$g_i(X) \leq 1 + (1-\alpha_i)t_i, \quad i = 1, \ldots, N'$

$g_i(X) \leq 1 + \gamma_i t_i, \quad i = 1, \ldots, N'$

$g_i(X) \leq 1 + \beta_i t_i, \quad i = 1, \ldots, N'$

$g_i(X) \geq 1 + (1-\alpha_i)t_i, \quad i = N'+1, \ldots, N$

$g_i(X) \geq 1 + \gamma_i t_i, \quad i = N'+1, \ldots, N$

$g_i(X) \geq 1 + \beta_i t_i, \quad i = N'+1, \ldots, N$

$X > 0, \alpha_i, \beta_i, \gamma_i \in (0,1], \quad i = 0, \ldots, N$

$\alpha \geq \gamma, \alpha \geq \beta, \alpha + \beta + \gamma \leq 3$

VI. The Concept of the Two-phase Method and the Proposed Algorithm

The overall process of optimization is divided into two phases which are described as follows:

Phase 1: In this phase first an appropriate GP problem is created for solving. Theorems 2 and 3 are then used to generate Single-valued Pentagonal Neutrosophic Linear Programming (SvPNLP) problem from the GP problem. The score function is then used to transform the SvPNLP problem into a crisp linear programming problem that allows the tolerance value to be set. In this case, the decision-maker implements his requirement according to his satisfaction. The decision-maker could choose various degrees of tolerance value, which results in distinct sets of feasible alternatives; consequently, we must devise a technique to determine the optimal solution within these feasible choices.

Phase 2: This phase begins with a feasible solution provided in phase 1 and its goal is to increase satisfaction by providing an optimal solution. The multi-parametric confidence vectors $\alpha, \beta, \gamma \in (0,1]^{(N+1)}$ are utilized to correlate the degree of satisfaction, dissatisfaction and indeterminacy with its relevant environment. Then the conversion of the objective function into a constraint takes place at this stage where $\mathcal{Z}_0$ marks the beginning of the optimum solution along with α_0, β_0 and γ_0 as satisfaction, dissatisfaction and hesitation degrees. The tolerance degree, t_i, can be enhanced for individual constraint and objective function, allowing the degree of satisfaction to be maximized and dissatisfaction degree to be minimized while maintaining the degree of indeterminacy in individual constraint. Finally, solving the original problem with the proposed model an optimal solution is achieved with the highest degree of satisfaction while keeping the level of dissatisfaction and indeterminacy in control.

An algorithm along with a flowchart, in Fig. 2 is presented for finding an optimal solution for SvPNGP problem based on the preceding discussion (3).

Algorithm: SvPNGP Modelling

1. Model the SvPNGP problem.

2 Convert the SvPNGP to crisp LP using the help of the score function and applying Theorem 2 and 3.

3 Find the initial optimal value $\mathcal{Z}_0$ from basic variables.

4. Add tolerance value and apply α, β - efficiency and formulate the equivalent LPP:

$$\max\ \alpha, \min \beta, \min \gamma \tag{15}$$

s.t $\mu_0(X) \geq \alpha,\ \vartheta_0(X) \leq \beta,\ \sigma_0(X) \leq \gamma$

$\mu_i(X) \geq \alpha,\ \vartheta_i(X) \leq \beta,\ \sigma_i(X) \leq \gamma,\ i = 1, \ldots, N,$

$\alpha \geq \beta, \alpha \geq \gamma, \alpha + \beta + \gamma \leq 3,$

$X > 0, \alpha_i, \beta_i, \gamma_i \in (0,1]$

The above LPP is equivalent to:

$$\text{s.t} \quad \max\ (\alpha - \beta - \gamma) \tag{16}$$

$\mu_0(X) \geq \alpha,\ \vartheta_0(X) \leq \beta,\ \sigma_0(X) \leq \gamma$

$\mu_i(X) \geq \alpha,\ \vartheta_i(X) \leq \beta,\ \sigma_i(X) \leq \gamma,\ i = 1, \ldots, N,$

$\alpha \geq \beta, \alpha \geq \gamma, \alpha + \beta + \gamma \leq 3,$

$X > 0, \alpha_i, \beta_i, \gamma_i \in (0,1]$

5. According to multi-parametric $\alpha, \beta, \gamma \in (0,1]^{(N+1)}$ apply the membership, non – membership and hesitancy functions and place the objective function as a constraint.

6. Solve and find $g_0(\mathcal{Z}^*,\alpha,\beta,\gamma)$ using the dual-simplex method.
7. Build a new programming problem model under multi-parametric α_i,β_i and γ_i with different degrees of satisfaction, dissatisfaction and indeterminacy respectively.
8. Solve the new problem and find the optimal satisfaction degree.
9. Determine the optimal value under optimal $\alpha^*,\beta^*,\gamma^*$ and evaluate $g_0(\mathcal{Z}^{**},\alpha^*,\beta^*,\gamma^*)$.

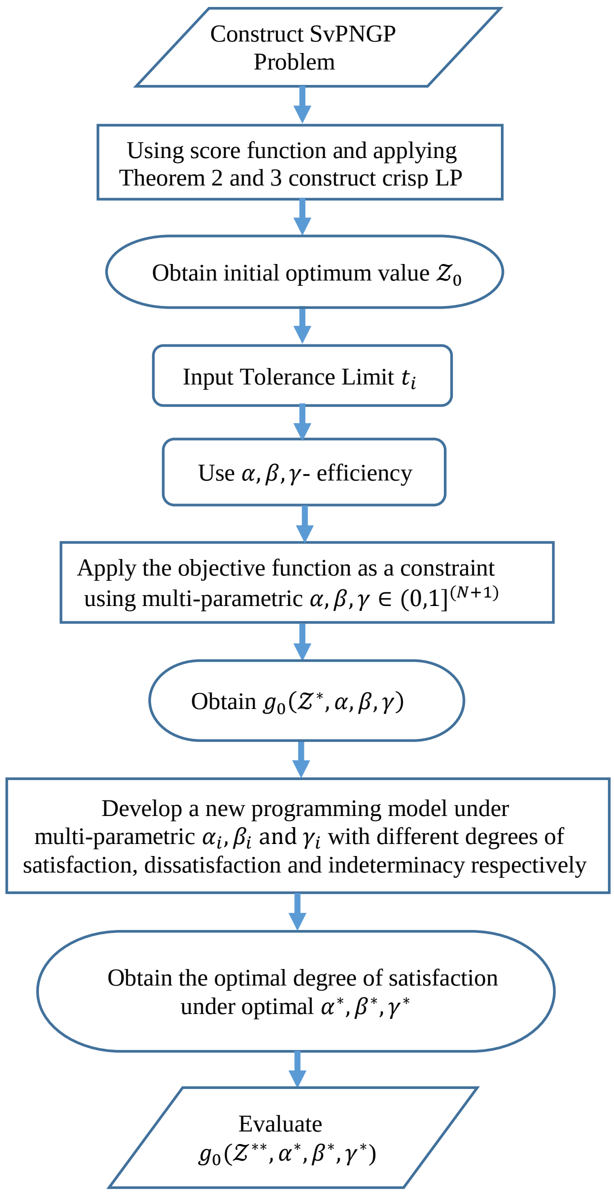

Fig. 2. Flowchart of the proposed work.

VII. Implementation of the Proposed Model with the Help of Numerical Illustration

Example 1. A water distribution plant wants to produce concrete pipes for an underground water distribution project. It requires three pipes P1, P2 and P3 with utmost weight e^{19}, e^{46}, and e^{64} in Kgs. The water pipes need to be manufactured utilizing four varieties of concrete materials M1, M2, M3 and M4. Table I shows the percentage of each kind of raw concrete required in each pipe (kg) and its unit price ($/kg). Determine the maximum amount of raw concrete required while staying within the owner's tolerance limit.

TABLE I. Concrete Percentages and its Price/Unit

Pipes	M1	M2	M3	M4	Need (Kg)
P1	P1M1	P1M2	P1M3	P1M4	e^{19}
P2	P2M1	P2M2	P2M3	P2M4	e^{46}
P3	P3M1	P3M2	P3M3	P3M4	e^{64}
Unit Price ($/Kg)	**5**	**6**	**3**	**5**	

P1M1= ((0,1,1,2,2);0.6,0.4,0.3), P1M2= ((0,1,3,4,5);0.9,0.1,0.3), P1M3= ((1,1,1,1,1);0.9,0.3,0.1), P1M4= ((1,2,2,3,4);0.8,0.5,0.3), P2M1= ((5,6,6,7,8);0.8,0.4,0.4), P2M2= ((3,4,6,7,9);0.8,0.5,0.3), P2M3= ((2,3,3,4,5);0.6,0.5,0.6), P2M4= ((0,2,2,4,5);0.8,0.2,0.5), P3M1= ((1,2,4,5,6);0.7,0.2,0.2), P3M2= ((2,3,5,6,8);0.7,0.2,0.2), P3M3= ((1,1,3,3,3);0.7,0.4,0.3), P3M4= ((10,11,13,14,15);0.8,0.4,0.2)

Solution. The above problem can be converted into SvPNGP as follows:

$$\widetilde{\max}\ x_1^5x_2^6x_3^3x_4^5 \tag{17}$$

$$\text{s.t}\quad x_1^{P1M1}x_2^{P1M2}x_3^{P1M3}x_4^{P1M4}, \leq e^{19}$$

$$x_1^{P2M1}x_2^{P2M2}x_3^{P2M3}x_4^{P2M4}, \leq e^{46}$$

$$x_1^{P3M1}x_2^{P3M2}x_3^{P3M3}x_4^{P3M4}, \leq e^{64}$$

$$x_1,x_2,x_3,x_4 > 0$$

By using x_i= $e^{\mathcal{Z}_j}$ $(1 \leq j \leq 4)$, we can change problem (16) into the intuitionistic fuzzy problem by utilizing Theorems 2 and 3.

$$\widetilde{\max}\ 5\mathcal{Z}_1 + 6\mathcal{Z}_2 + 3\mathcal{Z}_3 + 5\mathcal{Z}_4 \tag{18}$$

$$\text{s.t}\quad P1M1\mathcal{Z}_1 + P1M2\mathcal{Z}_2 + P1M3\mathcal{Z}_3 + P1M4\mathcal{Z}_4 \leq 19$$

$$P2M1\mathcal{Z}_1 + P2M2\mathcal{Z}_2 + P2M3\mathcal{Z}_3 + P2M4\mathcal{Z}_4 \leq 46$$

$$P3M1\mathcal{Z}_1 + P3M2\mathcal{Z}_2 + P3M3\mathcal{Z}_3 + P3M4\mathcal{Z}_4 \leq 64$$

Next, we apply the score function on SvPNN

$$\widetilde{\max}\ 5\mathcal{Z}_1 + 6\mathcal{Z}_2 + 3\mathcal{Z}_3 + 5\mathcal{Z}_4 \tag{19}$$

$$\text{s.t}\quad 0.76\mathcal{Z}_1 + 2.17\mathcal{Z}_2 + 0.83\mathcal{Z}_3 + 1.60\mathcal{Z}_4 \leq 19$$

$$4.27\mathcal{Z}_1 + 3.87\mathcal{Z}_2 + 1.70\mathcal{Z}_3 + 1.82\mathcal{Z}_4 \leq 46$$

$$2.64\mathcal{Z}_1 + 3.68\mathcal{Z}_2 + 1.47\mathcal{Z}_3 + 9.24\mathcal{Z}_4 \leq 64$$

After the conversion, the primary optimal solution is drawn from the basic variables $x_1 = e^{4.42}, x_2 = 1, x_3 = e^{11.40}, x_4 = e^{4.17}$ and the optimal value is $e^{76.18}$. By applying x_0= $e^{\mathcal{Z}_0}$, we get $\mathcal{Z}_0$= 76.18. Using the two-phase technique and applying the membership, non-membership and indeterminacy functions defined in (9) along with substituting the values of $\mathcal{Z}_0$, t_0, t_1, t_2 and t_3 where $t_0 = 5$, $t_1 = 1$, $t_2 = 4$ and $t_3 = 6$ for t_i, $(i = 0,1,2,3)$ are the tolerance values which are set up by the decision maker, we can convert problem (18) into the programming problem as given below:

$$\widetilde{\max}\,(\alpha - \beta - \gamma) \tag{20}$$

s.t. $5\mathcal{Z}_1 + 6\mathcal{Z}_2 + 3\mathcal{Z}_3 + 5\mathcal{Z}_4 \geq 76.18 - 5\alpha_0,$
$5\mathcal{Z}_1 + 6\mathcal{Z}_2 + 3\mathcal{Z}_3 + 5\mathcal{Z}_4 \geq 71.18 - 5\gamma_0,$
$5\mathcal{Z}_1 + 6\mathcal{Z}_2 + 3\mathcal{Z}_3 + 5\mathcal{Z}_4 \geq 71.18 - 5\beta_0,$
$0.76\mathcal{Z}_1 + 2.17\mathcal{Z}_2 + 0.83\mathcal{Z}_3 + 1.60\mathcal{Z}_4 \leq 20 - \alpha_1,$
$0.76\mathcal{Z}_1 + 2.17\mathcal{Z}_2 + 0.83\mathcal{Z}_3 + 1.60\mathcal{Z}_4 \leq 19 + \gamma_1,$
$0.76\mathcal{Z}_1 + 2.17\mathcal{Z}_2 + 0.83\mathcal{Z}_3 + 1.60\mathcal{Z}_4 \leq 19 + \beta_1,$
$4.27\mathcal{Z}_1 + 3.87\mathcal{Z}_2 + 1.70\mathcal{Z}_3 + 1.82\mathcal{Z}_4 \leq 50 - 4\alpha_2,$
$4.27\mathcal{Z}_1 + 3.87\mathcal{Z}_2 + 1.70\mathcal{Z}_3 + 1.82\mathcal{Z}_4 \leq 46 + 4\gamma_2,$
$4.27\mathcal{Z}_1 + 3.87\mathcal{Z}_2 + 1.70\mathcal{Z}_3 + 1.82\mathcal{Z}_4 \leq 46 + 4\beta_2,$
$2.64\mathcal{Z}_1 + 3.68\mathcal{Z}_2 + 1.47\mathcal{Z}_3 + 9.24\mathcal{Z}_4 \leq 70 - 6\alpha_3,$
$2.64\mathcal{Z}_1 + 3.68\mathcal{Z}_2 + 1.47\mathcal{Z}_3 + 9.24\mathcal{Z}_4 \leq 64 + 6\gamma_3,$
$2.64\mathcal{Z}_1 + 3.68\mathcal{Z}_2 + 1.47\mathcal{Z}_3 + 9.24\mathcal{Z}_4 \leq 64 + 6\beta_3,$
$\alpha_i, \beta_i, \gamma_i \in (0,1], \alpha_i \geq \beta_i, \alpha_i \geq \gamma_i, \alpha_i + \beta_i + \gamma_i \leq 3,$
$i = 0,1,2,3$

Table II displays the satisfaction of the decision-maker at various degrees of α, β, γ -efficiency confidence. If $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3), \beta = (\beta_0, \beta_1, \beta_2, \beta_3)$, $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ and $g_0(\mathcal{Z}^*, \alpha, \beta, \gamma)$ signify the optimal value of the objective function at every step under different conditions, we may obtain the following table using the LINGO 18.0 software:

From Table II it can be observed that the maximum initial optimal solution $g_0(\mathcal{Z}, \alpha_i, \beta_i, \gamma_i)$ is achieved at row 1 with a value of 80.11. It is also seen that the least efficient components are α_1 and α_2 . As the values of α_1 and α_2 increases, the values of $\gamma_1, \gamma_2, \beta_1$ and β_2 decreases because of the constraint $\alpha_i \geq \beta_i$ and $\alpha_i \geq \gamma_i$, that results in the degradation of the optimal solution. By reducing the values of α_1 and α_2 provides better results. If we increase the values of α_0 and α_3 and decrease γ_0, γ_3, β_0 and β_3 respectively, we are able to reach closer to the optimal solution. Now, as an initial solution, we will strive to minimize α_1 and α_2 by selecting the (0.5, 0.5, 0.5, 0.5)-efficient solution having optimal value $g_0(\mathcal{Z}, \alpha_i, \beta_i, \gamma_i)$ = 80.11.

Now, we will determine the LP problem below that is influenced by $g_0(\mathcal{Z}, \alpha_i, \beta_i, \gamma_i)$ = 80.11.

$$\widetilde{\max} \sum_{i=0}^{3} (\alpha_i - \beta_i - \gamma_i) \quad (21)$$

s.t. $5\mathcal{Z}_1 + 6\mathcal{Z}_2 + 3\mathcal{Z}_3 + 5\mathcal{Z}_4 \geq 80.11 - 5\alpha_0,$
$5\mathcal{Z}_1 + 6\mathcal{Z}_2 + 3\mathcal{Z}_3 + 5\mathcal{Z}_4 \geq 75.11 - 5\gamma_0,$
$5\mathcal{Z}_1 + 6\mathcal{Z}_2 + 3\mathcal{Z}_3 + 5\mathcal{Z}_4 \geq 75.11 - 5\beta_0,$
$0.76\mathcal{Z}_1 + 2.17\mathcal{Z}_2 + 0.83\mathcal{Z}_3 + 1.60\mathcal{Z}_4 \leq 20 - \alpha_1,$
$0.76\mathcal{Z}_1 + 2.17\mathcal{Z}_2 + 0.83\mathcal{Z}_3 + 1.60\mathcal{Z}_4 \leq 19 + \gamma_1,$
$0.76\mathcal{Z}_1 + 2.17\mathcal{Z}_2 + 0.83\mathcal{Z}_3 + 1.60\mathcal{Z}_4 \leq 19 + \beta_1,$
$4.27\mathcal{Z}_1 + 3.87\mathcal{Z}_2 + 1.70\mathcal{Z}_3 + 1.82\mathcal{Z}_4 \leq 50 - 4\alpha_2,$
$4.27\mathcal{Z}_1 + 3.87\mathcal{Z}_2 + 1.70\mathcal{Z}_3 + 1.82\mathcal{Z}_4 \leq 46 + 4\gamma_2,$
$4.27\mathcal{Z}_1 + 3.87\mathcal{Z}_2 + 1.70\mathcal{Z}_3 + 1.82\mathcal{Z}_4 \leq 46 + 4\beta_2,$
$2.64\mathcal{Z}_1 + 3.68\mathcal{Z}_2 + 1.47\mathcal{Z}_3 + 9.24\mathcal{Z}_4 \leq 70 - 6\alpha_3,$
$2.64\mathcal{Z}_1 + 3.68\mathcal{Z}_2 + 1.47\mathcal{Z}_3 + 9.24\mathcal{Z}_4 \leq 64 + 6\gamma_3,$
$2.64\mathcal{Z}_1 + 3.68\mathcal{Z}_2 + 1.47\mathcal{Z}_3 + 9.24\mathcal{Z}_4 \leq 64 + 6\beta_3,$
$0.5 \leq \alpha_0 \leq 1, 0.5 \leq \beta_0 \leq 1, 0.5 \leq \gamma_0 \leq 1$
$0.5 \leq \alpha_1 \leq 1, 0.5 \leq \beta_1 \leq 1, 0.5 \leq \gamma_1 \leq 1$
$0.5 \leq \alpha_2 \leq 1, 0.5 \leq \beta_2 \leq 1, 0.5 \leq \gamma_2 \leq 1$
$0.5 \leq \alpha_3 \leq 1, 0.5 \leq \beta_3 \leq 1, 0.5 \leq \gamma_3 \leq 1$

The optimum solution of problem (20) will be reached by maximizing the satisfaction degree as $\mathcal{Z}^{**} = (5.64,0,10.76,3.93)$ with confidence level $\alpha^* = (1, 0.5, 0.5, 0.5)$, $\beta^* = (0, 0.5, 0.5, 0.5)$ and $\gamma^* = (0, 0.5, 0.5, 0.5,$ and the optimal value calculated with respect to α^*, β^* and γ^* as $g_0(\mathcal{Z}^{**}, \alpha^*, \beta^*, \gamma^*) =$ (5.64,0,10.76,3.93;1,0.5,0.5,0.5;0,0.5,0.5,0.5;0,0.5,0.5,0.5)=80.11 . As a result, the optimal solution to GP programming problem (16) is $x_1 = e^{5.64}$, x_2 =1, $x_3 = e^{10.76}$, $x_4 = e^{3.93}$, and the optimal value is $e^{80.11}$.

TABLE II. DETERMINING THE MAXIMUM LEVEL OF SATISFACTION WITH A MULTI-PARAMETERS α, β, γ

S. No	$\alpha_0, \alpha_1, \alpha_2, \alpha_3$	$\gamma_0, \gamma_1, \gamma_2, \gamma_3$	$\beta_0, \beta_1, \beta_2, \beta_3$	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$g_0(\mathcal{Z}, \alpha_i, \beta_i, \gamma_i)$
1	0.5,0.5,0.5,0.5	0.5,0.5,0.5,0.5	0.5,0.5,0.5,0.5	5.64	0	10.76	3.93	80.11
2	0.9,0.5,0.5,0.7	0.2,0.5,0.5,0.1	0.1,0.5,0.5,0.2	5.44	0	11.58	3.59	79.93
3	0.4,0.8,0.3,1	0.2,0.4,0.2,0.3	0.1,0.5,0.2,0.6	5.22	0	11.36	3.63	78.32
4	0.5,0.9,0.9,0.5	0,0.3,0.2,0.1	0.3,0.4,0.5,0.3	5.36	0	10.93	3.72	78.20
5	0.5,0.8,0.9,0.5	0.2,0.1,0.2,0.3	0.1,0.3,0.1,0.2	5.27	0	10.80	3.83	77.89
6	0.7,0.5,0.8,0.5	0.7,0.5,0.8,0.5	0.2,0.5,0.1,0.5	5.21	0	11.00	3.75	77.85
7	0.9,0.9,0.9,0.9	0,0,0,0	0.3,0.2,0.2,0.3	5.11	0	11.07	3.70	77.28

Example 2. In continuation from example 1, determining the maximum satisfaction of decision maker without any tolerance limit then the results are shown in Table III.

TABLE III. DETERMINING THE MAXIMUM SATIFACTION LEVEL WITHOUT TOLERANCE LIMIT

S. No	1	2	3	4
α	0.5,0.5,0.5,0.5	0.9,0.5,0.5,0.7	0.7,0.5,0.8,0.5	0.5,0.9,0.9,0.5
γ	0.5,0.5,0.5,0.5	0.2,0.5,0.5,0.1	0.2,0.1,0.3,0.2	0,0.3,0.2,0.1
β	0.5,0.5,0.5,0.5	0.1,0.5,0.5,0.2	0.2,0.5,0.1,0.5	0.3,0.4,0.5,0.3
$\mathcal{Z}_1$	4.89	4.85	5.07	5.06
$\mathcal{Z}_2$	0	0	0	0
$\mathcal{Z}_3$	11.90	12.04	11.20	11.23
$\mathcal{Z}_4$	3.69	3.63	3.71	3.70
G_0	78.59	78.56	77.55	77.54

$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3), \beta = (\beta_0, \beta_1, \beta_2, \beta_3), \gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3), G_0 = g_0(\mathcal{Z}, \alpha_i, \beta_i, \gamma_i)$

Example 3. With continuation from example 1, determining the maximum satisfaction of decision maker with single parametric α, β and γ then the results are displayed in Table IV.

TABLE IV. DETERMINING THE MAXIMUM SATIFACTION LEVEL WITH SINGLE PARAMETRIC α, β AND γ

S. No	α	γ	β	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	G_0
1	0.5	0.5	0.5	5.64	0	10.76	3.93	80.11
2	0.6	0.4	0.4	5.53	0	10.82	3.88	79.55
3	0.8	0.2	0.2	5.32	0	10.94	3.79	78.41
4	0.9	0.1	0.1	5.21	0	11.00	3.74	77.85
5	1	0	0	5.10	0	11.07	3.70	77.28

$G_0 = g_0(\mathcal{Z}, \alpha_i, \beta_i, \gamma_i)$

It can be observed that the optimal solution degrades with the absence of tolerance limit while analyzing Table II and III. When Table II and IV are compared, it is found that in Table IV, raising the confidence level α and decreasing γ and β, reduces the ideal solution, whereas we anticipate the optimal solution to increase as the confidence level rises. Similarly, the decision maker has to adjust the levels of satisfaction, dissatisfaction and hesitancy levels for every constraint to the same degree, but in the method proposed, the satisfaction, dissatisfaction and hesitancy levels for each constraint can be decreased or increased independently. Table IV clearly shows that raising the confidence level has the opposite effect on the optimization, and the optimal value returns to its original solution, whereas in Table II, until the satisfaction, dissatisfaction and hesitancy degree components change, the optimal value remains optimal. The reason for this is because the confidence vectors are not reasonable for all constraints, especially when the objective function is transformed into a constraint. Thus the adaptability of the $(N + 1)$ -dimensional α_i, β_i and γ_i confidence levels can help in achieving the decision-maker's purpose of getting a better optimal result.

Table V presents the difference of solutions for example 1 using four methods. Comparing our work with Intuitionistic Fuzzy Geometric Programming (IFGP), Khorsandi et al., [36] and Zimmermann's method [37], it is observed that the solution achieved using our proposed method for solving SvPNGP is more efficient compared to the solution obtained using the other techniques. Here methods 1, 2 and 3 are multi-parametric, whereas method 4 is single parametric, and methods 3 and 4 are designed to solve fuzzy optimization problems, whereas Method 1 is intended to address Neutrosophic optimization problems.

The proposed method achieved the highest optimal value compared to the existing techniques. Fuzzy optimization only considers one degree of acceptance or rejection at a time whereas Intuitionistic Fuzzy optimization includes both degrees of acceptance and rejection in order to manage optimization but in reality, there are some circumstances where, due to lack of information or indeterminacy, evaluating the membership and non-membership functions together cannot yield a greater and/or more satisfactory conclusion. As a result, there is still an indeterministic element on which hesitation persists which is addressed by neutrosophic optimization.

TABLE V. COMPARATIVE ANALYSIS BETWEEN DIFFERENT OPTIMIZATION APPROACHES

Methods	α	β	γ	t_0	t_1	t_2	t_3	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	G_0
1) Proposed SvPNGP	(1, 0.5, 0.5, 0.5)	(0, 0.5, 0.5, 0.5)	(0, 0.5, 0.5, 0.5)	5	1	4	6	5.64	0	10.76	3.93	80.11
2) IFGP	(1, 0.5, 0.5, 0.5)	(0, 0.5, 0.5, 0.5)		5	1	4	6	0	8.52	0	6.99	76.78
3) Khorsandi et al's Method [36]	(1, 0.5, 0.5, 0.5)			5	1	4	6	2.51	5.3	0	3.18	60.34
4) Zimmermann's Method [37]	(0.2) (0.5) (1)			5	1	4	6	2.74 2.51 2.12	5.21 5.30 5.45	0 0 0	3.30 3.18 2.98	61.59 60.34 58.27

$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3), \beta = (\beta_0, \beta_1, \beta_2, \beta_3), \gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3), G_0 = g_0(\mathcal{Z}, \alpha_i, \beta_i, \gamma_i)$

VIII. CONCLUSION

In order to obtain optimal results in decision making, the decision-maker needs to be provided with the flexibility to achieve satisfaction in the decision making process. Thereby, this research article proposed a SvPNGP model by incorporating multi-parametric vectors α, β, γ to achieve the maximum degree of satisfaction while minimizing the degree of dissatisfaction and hesitation within the tolerance limit of the decision-maker. With this strategy, the decision-maker can obtain an optimal solution for the SvPNGP problem while satisfying his/her needs and moreover the decision maker is not restricted for selecting the same tolerance value for individual constraints. We divided the whole process into a two-phase method where the SvPNGP is transformed to a crisp LP problem in the first phase and in the next phase, the multi-parametric vectors are applied along with membership, indeterminacy and non-membership functions and solved to find the optimal solution. With the help of numerical problems, we evaluated and analyzed certain parameters with our proposed model. The results were then compared with the existing methods and found out to produce better optimal solution compared to others.

The contribution of this paper includes developing an optimal SvPNGP model to enable the decision-maker to achieve robust decisions while providing him the flexibility to achieve the desired level of satisfaction. As the model is built on neutrosophic numbers, it can handle uncertainty in real-world programming situations.

For future work, we hope to expand our work with Plithogenic sets, which is another generalized method that can be useful for dealing with inconsistent and indeterminate data. The extended approach can be used to a wide range of real-world challenges in the field of engineering, manufacturing, management and many more.

REFERENCES

[1] M. Abdallah, S. Hamdan, and A. Shabib, "A multi-objective optimization model for strategic waste management master plans," Journal of Cleaner Production, 284, 2021, p.124714. https://doi.org/10.1016/j.jclepro.2020.124714.

[2] P. Goudarzi, A.M Rahmani, and M. Mosleh, "A mathematical optimization model using red deer algorithm for resource discovery in CloudIoT," Transactions on Emerging Telecommunications Technologies, 33(12), 2022, p.e4646. https://doi.org/10.1002/ett.4646.

[3] S. Nath, P. Das, and P. Debnath, "A Brief Review on Multi-Attribute Decision Making in the Emerging Fields of Computer Science," In Computational Intelligence in Communications and Business Analytics: 4th International Conference, CICBA 2022, Silchar, India, Revised Selected Papers, Cham: Springer International Publishing, January 7–8, 2022, pp. 3-18. https://doi.org/10.1007/978-3-031-10766-5_1.

[4] S. Pant, P. Garg, A. Kumar, M. Ram, A. Kumar, H. K. Sharma, and Y. Klochkov, "AHP-based multi-criteria decision-making approach for monitoring health management practices in smart healthcare system," International Journal of System Assurance Engineering and Management, 2023, pp.1-12. https://doi.org/10.1007/s13198-023-01904-5.

[5] M.B. Bouraima, Y. Qiu, Ž. Stević, and V. Simić, "Assessment of alternative railway systems for sustainable transportation using an integrated IRN SWARA and IRN CoCoSo model," Socio-Economic Planning Sciences, 86, 2023, p.101475. https://doi.org/10.1016/j.seps.2022.101475.

[6] C.H. Huang, "Engineering design by geometric programming," Mathematical problems in engineering, 2013. https://doi.org/10.1155/2013/568098.

[7] R.J. Duffin, "Geometric programming-theory and application," vol. QA264, 1967.

[8] L.A. Zadeh, "Fuzzy sets," Information and control,8(3), 1965,pp. 338-353.

[9] R.E. Bellman, and L.A. Zadeh, "Decision-making in a fuzzy environment," Management science, 17(4), 1970, pp.B-141. https://doi.org/10.1287/mnsc.17.4.B141.

[10] H. Tanaka, T. Okuda, and K. Asai, "Fuzzy mathematical programming," Transactions of the society of instrument and control engineers, 9(5), 1973, pp.607-613. https://doi.org/10.9746/sicetr1965.9.607.

[11] H.J. Zimmermann, "Fuzzy programming and linear programming with several objective functions," Fuzzy sets and systems, 1(1), 1978, pp.45-55. https://doi.org/10.1016/0165-0114(78)90031-3.

[12] B.S. Mahapatra, and G.S. Mahapatra, "Reliability and cost analysis of series system models using fuzzy parametric geometric programming," Fuzzy Information and Engineering, 2(4), 2010, pp.399-411. https://doi.org/10.1007/s12543-010-0058-1.

[13] F. Samadi, A. Mirzazadeh, and M.M. Pedram, "Fuzzy pricing, marketing and service planning in a fuzzy inventory model: A geometric programming approach," Applied Mathematical Modelling, 37(10-11), 2013, pp.6683-6694. https://doi.org/10.1016/j.apm.2012.12.020.

[14] K. T. Atanassov, "Intuitionistic fuzzy sets," Fuzzy sets and Systems, 20(1), 1986,pp. 87-96. https://doi.org/10.1007/978-3-7908-1870-3_1.

[15] E. Jafarian, J. Razmi, and M.F. Baki, "A flexible programming approach based on intuitionistic fuzzy optimization and geometric programming for solving multi-objective nonlinear programming problems," Expert Systems with Applications, 93, 2018, pp.245-256. https://doi.org/10.1016/j.eswa.2017.10.030.

[16] Z. Kheiri, and B.Y. Cao, "Posynomial geometric programming with intuitionistic fuzzy coefficients," Fuzzy Systems & Operations Research and Management, Springer, Cham, 2016, pp. 15-30. https://doi.org/10.1007/978-3-319-19105-8_2.

[17] S. Islam, and W.A. Mandal, "Fuzzy geometric programming techniques and applications," Springer Singapore, 2019. https://doi.org/10.1007/978-981-13-5823-4.

[18] Y. Xue, and Y. Deng, "Decision making under measure-based granular uncertainty with intuitionistic fuzzy sets," Applied intelligence, 51, 2021, pp.6224-6233. https://doi.org/10.1007/s10489-021-02216-6

[19] F. Smarandache, "A unifying field in Logics: Neutrosophic Logic," Philosophy, American Research Press, 1999, pp. 1-141.

[20] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, "Single valued neutrosophic sets," Infinite study, 12, 2010.

[21] J.S. Chai, G. Selvachandran, F. Smarandache, V.C. Gerogiannis, L.H. Son, Q.T. Bui, and B. Vo, "New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems," Complex & Intelligent Systems, 7, 2021, pp.703-723. https://doi.org/10.1007/s40747-020-00220-w

[22] V. Başhan, H. Demirel, and M. Gul, "An FMEA-based TOPSIS approach under single valued neutrosophic sets for maritime risk evaluation: the case of ship navigation safety," Soft Computing, 24(24), 2020, pp.18749-18764. https://doi.org/10.1007/s00500-020-05108-y

[23] M. Luo, G. Zhang, and L. Wu, "A novel distance between single valued neutrosophic sets and its application in pattern recognition," Soft Computing, 26(21), 2022, pp.11129-11137. https://doi.org/10.1007/s00500-022-07407-y

[24] M. Ali, Z. Hussain, and M.S. Yang, "Hausdorff Distance and Similarity Measures for Single-Valued Neutrosophic Sets with Application in Multi-Criteria Decision Making," Electronics, 12(1), 2022, p.201. https://doi.org/10.3390/electronics12010201

[25] M. Abdel-Basset, M. Mohamed, A.N. Hussien and A.K. Sangaiah, "A novel group decision-making model based on triangular neutrosophic numbers," Soft Comput, 22(20), 2018, pp. 6629–6643. https://doi.org/10.1007/s00500-017-2758-5

[26] J. Ye, "Trapezoidal neutrosophic set and its application to multiple attribute decision-making," Neural Comput Appl, 26(5), 2015, pp.1157–1166. https://doi.org/10.1007/s00521-014-1787-6.

[27] S.K. Das, and A. Chakraborty, "A new approach to evaluate linear programming problem in pentagonal neutrosophic environment," Complex & intelligent systems, 7, 2021, pp.101-110. https://doi.org/10.1007/s40747-020-00181-0.

[28] H.A.E.W. Khalifa, M. Saeed, A.U. Rahman, and S. El-Morsy, "An Application of Pentagonal Neutrosophic Linear Programming for Stock Portfolio Optimization," Neutrosophic Sets and Systems, 51(1), 2022, p.41. . https://digitalrepository.unm.edu/nss_journal/vol51/iss1/41.

[29] C. Kar, T.K. Roy, and M. Maiti, "EOQ model with price, marketing, service and green dependent neutrosophic demand under uncertain resource constraint: A geometric programming approach," Neutrosophic Sets and Systems, 51, 2022, pp.797-823. http://fs.unm.edu/NSS2/index.php/111/article/view/2601.

[30] B.Y. Cao, "Optimal models and methods with fuzzy quantities," vol. 248, 2010, Berlin: Springer. https://doi.org/10.1007/978-3-642-10712-2.

[31] H. Attari, and S.H. Nasseri, "New concepts of feasibility and efficiency of solutions in fuzzy mathematical programming problems," Fuzzy Information and Engineering, 6(2), 2014, pp.203-221. https://doi.org/10.1016/j.fiae.2014.08.005.

[32] A. Chakraborty, S. Broumi and P.K. Singh, "Some properties of pentagonal neutrosophic numbers and its applications in transportation problem environment," Infinite Study, 2019. https://digitalrepository.unm.edu/nss_journal/vol28/iss1/16.

[33] S.T. Liu, "Geometric programming with fuzzy parameters in engineering optimization. International Journal of Approximate Reasoning," 46(3), 2007, pp.484-498. https://doi.org/10.1016/j.ijar.2007.01.004.

[34] B.Y. Cao, and J.H. Yang, "Advances in fuzzy geometric programming," In Fuzzy Information and Engineering: Proceedings of the Second International Conference of Fuzzy Information and Engineering (ICFIE), Springer Berlin Heidelberg, 2007, pp. 497-502. https://doi.org/10.1007/978-3-540-71441-5.

[35] M. Khan, M. Zeeshan and S. Iqbal, "Neutrosophic variational inequalities with applications in decision-making," Soft Computing, 26(10), 2022, pp.4641-4652. https://doi.org/10.1007/s00500-022-06956-6.

[36] A. Khorsandi, B.Y. Cao, and H. Nasseri, "A New Method to Optimize the Satisfaction Level of the Decision-maker in Fuzzy Geometric Programming Problems," Mathematics, 7(5), 2019, p.464. https://doi.org/10.3390/math7050464.

[37] C.R. Bector and S. Chandra, "Fuzzy mathematical programming and fuzzy matrix games," Vol. 169, Berlin: Springer, 2005. https://doi.org/10.1007/3-540-32371-6.