Limit Cycle Generation for Multi-Modal and 2-Dimensional Piecewise Affine Control Systems

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Abstract—**This paper considers a limit cycle control problem of a multi-modal and 2-dimensional piecewise affine control system. Limit cycle control means a controller design method to generate a limit cycle for given piecewise affine control systems. First, we deal with a limit cycle synthesis problem and derive a new solution of the problem. In addition, theoretical analysis on the rotational direction and the period of a limit cycle is shown. Next, the limit cycle control problem for piecewise affine control system is formulated. Then, we obtain matching conditions such that the piecewise affine control system with the state feedback law corresponds to the reference system which generates a desired limit cycle. Finally, in order to indicate the effectiveness of the new method, a numerical simulation is illustrated.**

I. INTRODUCTION

Limit cycles are known to be quite important concept in various research fields [1]. We can find limit cycles in real world, for example, stable walking or gaits of humanoid robots in robotic engineering, oscillator circuits in electronic engineering, catalytic hypercycles in chemistry, circadian rhythms in biology, boom-bust cycles in economics and so on. Researches on limit cycles have been eagerly done from both mathematical and engineering perspectives so far [2], [3], [4], [5], [6], [7], [8], [9], [11], [12], [13], [14]. Especially, some conditions for nonlinear systems that generate periodic solutions and some applications were shown in [2], and in [7], a synthesis method of hybrid systems whose solution trajectories converge to desired trajectories was proposed. In these studies, it is guaranteed that solution trajectories of the systems converges to a desired closed curve, and the existence of limit cycles was confirmed by numerical simulations, However, the mathematical guarantee of the existence of limit cycles was not shown. On the other hand, the authors proposed a synthesis method of multi-modal and 2-dimensional piecewise affine systems that generate desired limit cycles in [10], [15] and showed a mathematical proof of the existence and the uniqueness of a limit cycle for the proposed system. In addition, some theoretical analysis on the rotational direction and the period of a limit cycle is derived. In this study, we assume that the whole of a system can be designed. A method to generate a desired limit cycle for a given piecewise affine control system with tuning some parameters of the system is more useful for a wide variety of situations. However, such a control method have not been proposed so far.

Hence, we consider a limit cycle control problem of multi-modal and 2-dimensional piecewise affine systems in this paper. The outline of this paper is as follows. We first consider a limit cycle synthesis problem and derive its new solution in Section II. Some theoretical properties are also shown. Then, in Section III, a formulation of limit cycle control problem is presented, and necessary and sufficient conditions for the problem, which are called *matching conditions*, are derived. Finally, a numerical simulation is shown in order to confirm the effectiveness of the new method in Section IV.

II. LIMIT CYCLE SYNTHESIS OF PIECEWISE AFFINE SYSTEMS

A. Formulation of Limit Cycle Synthesis

In this section, we consider a synthesis problem of piecewise affine systems which generate desired limit cycles. First, this subsection give the formulation of the problem. Consider the 2-dimensional Euclidian space: $\mathbf{R}^2$, its coordinate: $x = [\, x_1 \ x_2 \,]^\mathsf{T} \in \mathbf{R}^2$, and the origin of $\mathbf{R}^2$: O. Let us set $N \, (N \geq 3)$ points $P_i \neq O \, (i = 1, \cdots, N)$ in $\mathbf{R}^2$ and denote the vector from O to P_i by $p_i = [\, p_i^1 \ p_i^2 \,]^\mathsf{T}$. We also denote the angle between the half line OP_i and the x_1-axis by θ_i. Now, without loss of generality, we assume that the N points $P_i \, (i = 1, \cdots, N)$ are located in the counterclockwise rotation from the x_1-axis, that is, $0 \leq \theta_1 < \theta_2 < \cdots < \theta_N$ holds.

Next, we define the semi-infinite region D_i which is sandwiched by the half lines OP_i and OP_{i+1} and the line segment C_i joining P_i and P_{i+1}, where $P_{N+1} = P_1$. Set a polygon that is a union of $C_i \, (i = 1, \cdots, N)$ as

$$C := \bigcup_{i=1}^{N} C_i. \tag{1}$$

Fig. 1 shows an example of a Polygonal Closed Curve for $N = 5$.

We then consider the affine system defined in D_i:

$$\dot{x} = a_i + A_i x, \ \ x \in D_i \tag{2}$$

where $x = [\, x_1 \ x_2 \,]^\mathsf{T} \in \mathbf{R}^2$ is the state variable, and $a_i \in \mathbf{R}^2, \ A_i = \mathbf{R}^{2 \times 2}$ are the affine term and the coefficient matrix, respectively. Consequently, we treat the N-modal and 2-dimensional piecewise affine system that consists of N regions $D_i \, (i = 1, \cdots, N)$ and N affine systems (2). In this paper, we consider the following synthesis problem on limit cycles called "the limit cycle synthesis problem."
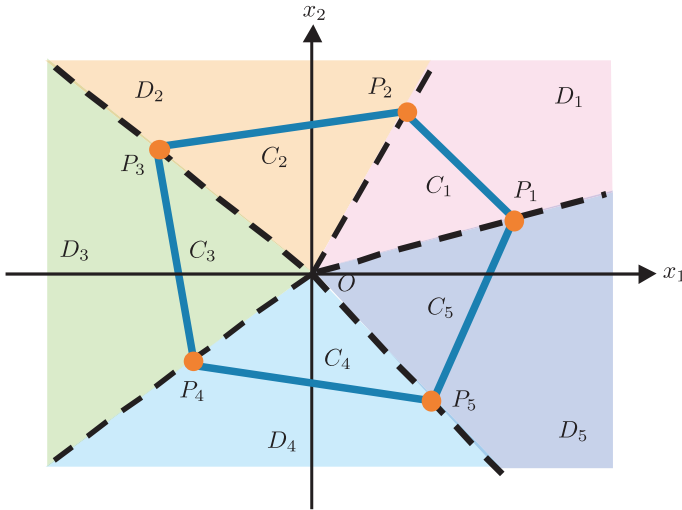
Fig. 1 : Example of Polygonal Closed Curve ($N = 5$)

Problem 1 : For the N-modal and 2-dimensional piecewise affine system (2), design a_i, A_i $(i = 1, \cdots, N)$ such that a given polygonal closed curve C (1) is a unique and stable limit cycle pf the system. □

A solution of Problem 1 has been derived in the author's previous studies [10], [15]. However, in this paper we will derive another solution which can be utilized to consider the limit cycle control problem shown in Section III.

B. Proposed System and Existence/Uniqueness of Limit Cycle

Next, we shall derive a solution for Problem 1 in this subsection. We also consider the existence and the uniqueness of a limit cycle for the obtained system. We focus on a behavior of a solution trajectory of (2) in D_i. It is easily confirmed that the equation of C_i is represented by

$$(p_i^2 - p_{i+1}^2)x_1 - (p_i^1 - p_{i+1}^1)x_2 \\ + p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1 = 0. \tag{3}$$

Using (3), we now define *a limit cycle function V_i* as

$$V_i(x) = (p_i^2 - p_{i+1}^2)x_1 - (p_i^1 - p_{i+1}^1)x_2 \\ + p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1. \tag{4}$$

If V_i converges to 0 along a solution trajectory of (2), then the solution trajectory of (2) also converges to C_i. Hence, a_i and A_i should be determined so that V_i converges to 0 along a solution trajectory of (2). Now, an important result on design of nonlinear systems is shown as follows [2].

Theorem 1 [2] : Consider the 2-dimensional nonlinear system:

$$\dot{x} = f(x) + g(x), \tag{5}$$

where $x \in \mathbf{R}^2$ and $f, g \in \mathbf{R}^2 \to \mathbf{R}^2$ are vector fields defined in $\mathbf{R}^2$. In addition, consider a radial and unbounded function define on $\mathbf{R}^2$: $V : \mathbf{R}^2 \to \mathbf{R}$ such that $V(0) = 0$ and $V(x) \neq 0$, $\forall x \neq 0$ hold. We now define f and g as

$$f := U_f(x)\frac{\partial V}{\partial x}^\mathsf{T}, \quad g := -u_g(V)U_g(x)\frac{\partial V}{\partial x}^\mathsf{T}, \tag{6}$$

where a skew-symmetric matrix U_f, a positive definite matrix U_g, and u_g such that $u_g(V) > 0$, $V \neq 0$ holds. Then, for the system (5) with (6),

$$\lim_{t \to \infty} V(x(t)) = 0 \tag{7}$$

holds □

Applying Theorem 1, we can derive an affine term a_i and a coefficient matrix A_i of the affine system (2) such that V_i converges to 0 along a solution trajectory of (2). As a specific form of (5) and (6), we use

$$\dot{x} = f_i + g_i, \\ f_i := \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} \frac{\partial V_i}{\partial x}^\mathsf{T}, \\ g_i := -V_i(x)\begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix} \frac{\partial V_i}{\partial x}^\mathsf{T}, \tag{8}$$

where ω_i and $\lambda_i > 0$ are design parameters. Substituting (4) into (8) and comparing it with (2), we can obtain a_i and A_i of (2) as

$$a_i = \\ \begin{bmatrix} -\lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i(p_i^1 - p_{i+1}^1) \\ \lambda_i(p_i^1 - p_{i+1}^1)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i(p_i^2 - p_{i+1}^2) \end{bmatrix}, \\ A_i = \\ \begin{bmatrix} -\lambda_i(p_i^2 - p_{i+1}^2)^2 & \lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) \\ \lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) & -\lambda_i(p_i^1 - p_{i+1}^1)^2 \end{bmatrix}. \tag{9}$$

Compared to the piecewise affine system shown in [10], [15], (28) contains a new parameter λ_i and this additional parameter plays an important role in the limit cycle control problem in Section III. It is noted that that the system (2) with (28) satisfies only the convergence property (7), that is, its solution trajectory converges to C_i in D_i. Hence, we will discuss the existence of a unique and stable limit cycle of the system (2) with (28). To prove this, we first indicate three lemmas, and then we show the main theorem by using them. Now, we give the definition on the clockwise and counterclockwise rotations of limit cycle solution trajectories of the system (2) with (28) [10], [15].

Definition 1 [10], [15] : For limit cycle solution trajectories of the N-modal and 2-dimensional piecewise affine system (2) with (28), one that rotates in the clockwise direction in $\mathbf{R}^2$ is called *a limit cycle solution trajectory in the clockwise rotation*. On the contrary, one that rotates in the counterclockwise direction in $\mathbf{R}^2$ is called *a limit cycle solution trajectory in the counterclockwise rotation* (see Fig. 2). □

Let us define a subset in D_i as

$$M_i(\varepsilon_i) := \{ x \in D_i \,|\, \varepsilon_i^- \leq V_i(x) \leq \varepsilon_i^+ \}, \tag{10}$$

where $\varepsilon_i^-, \varepsilon_i^+ \in \mathbf{R}$ satisfies $\varepsilon_i^- < \varepsilon_i^+$ and we set $\varepsilon_i = (\varepsilon_i^-, \varepsilon_i^+)$. We also define a sum of these subsets as

$$M(\varepsilon) := \bigcup_{i=1}^{N} M_i(\varepsilon_i), \tag{11}$$

where we use the notations: $\varepsilon^- = (\varepsilon_1^-, \cdots, \varepsilon_N^-)$, $\varepsilon^+ = (\varepsilon_1^+, \cdots, \varepsilon_N^+)$, $\varepsilon = (\varepsilon^-, \varepsilon^+)$, and the parameters ε^- and ε^+

are determined such that

$$M^-(\varepsilon^-) = \bigcup_{i=1}^{N} \{\, x \in D_i \mid V_i(x) = \varepsilon_i^- \,\}$$

$$M^+(\varepsilon^+) = \bigcup_{i=1}^{N} \{\, x \in D_i \mid V_i(x) = \varepsilon_i^+ \,\}$$

(12)

form closed polygons. If $M^-(\varepsilon^-)$ and $M^+(\varepsilon^+)$ are closed polygons, that is, $M(\varepsilon)$ is a bounded and closed set, then ε is said to be *admissible* (see Fig. 3). We can derive the following proposition on $M(\varepsilon)$.

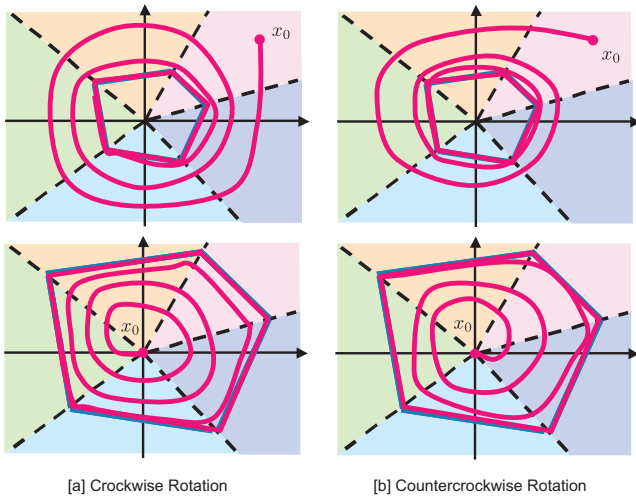[a] Crockwise Rotation   [b] Countercrockwise Rotation

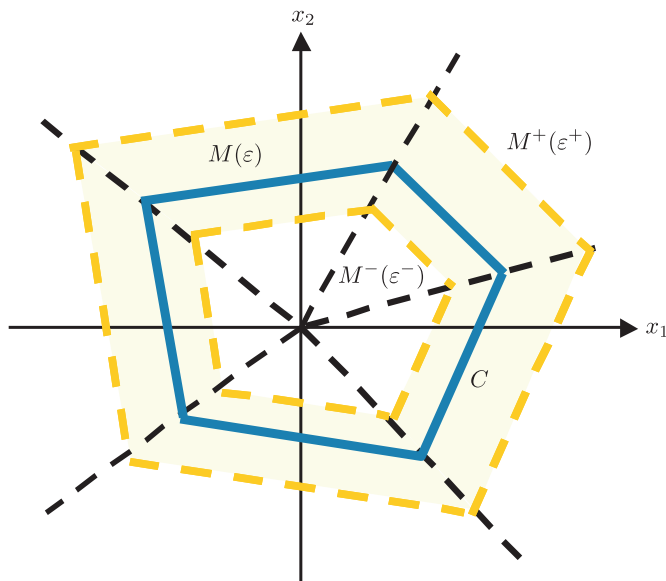Fig. 2 : Clockwise and Counterclockwise Rotations of Limit Cycle Solution Trajectories

Fig. 3 : Example of $M^-(\varepsilon^-)$, $M^+(\varepsilon^+)$ and $M(\varepsilon)$

Lemma 1 : For the N-modal and 2-dimensional piecewise affine system (2) with (28), $M(\varepsilon)$ is a positively invariant, bounded and closed set for any admissible ε.

(Proof) Calculating the time differential of $1/2V_i^2$ along a

solution trajectory of the system (2) with (28), we have

$$\frac{1}{2}\frac{d}{dt}(V_i^2) = V_i\dot{V}_i = V_i\frac{\partial V_i}{\partial x}\dot{x} = V_i\frac{\partial V_i}{\partial x}(f_i + g_i)$$

$$= V_i\frac{\partial V_i}{\partial x}\begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}\frac{\partial V_i}{\partial x}^{\mathsf{T}}$$

$$-V_i^2\frac{\partial V_i}{\partial x}\begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix}\frac{\partial V_i}{\partial x}^{\mathsf{T}}$$

$$= -\lambda_i^2 V_i^2\frac{\partial V_i}{\partial x}\frac{\partial V_i}{\partial x}^{\mathsf{T}} < 0.$$

Hence, it turns out that the velocity vector field of the system (2) with (28) points to the direction of the inner side of the bounded and closed set M at any points on the boundary $M^-(\varepsilon^-) \cup M^+(\varepsilon^+)$ of M. Consequently, $M(\varepsilon)$ is a positively invariant, bounded and closed set. $\square$

Next, we consider equilibrium points of the system (2) with (28). The following lemma on equilibrium points can be obtained.

Lemma 2 : Assume $\omega_i \neq 0$ $(i = 1, \cdots, N)$. Then, the N-modal and 2-dimensional piecewise affine system (2) with (28) does not have any equilibrium points in $M(\varepsilon)$ for any admissible ε.

(Proof) The unit vector which is on a parallel with C_i in D_i and points to the counterclockwise rotation is given by $(p_i - p_{i+1})/||p_i - p_{i+1}||$. By considering the inner product of this unit vector and the velocity vector field of the system (2) with (28), we have the magnitude of the velocity component to the direction of $p_i - p_{i+1}$ for a solution trajectory v_i of the system (2) with (28) in D_i as

$$v_i = (a_i + A_i x) \cdot \frac{p_i - p_{i+1}}{||p_i - p_{i+1}||}. \qquad (13)$$

Now, we denote a point in D_i by $x = \alpha_i p_i + \beta_i p_{i+1}, \alpha_i, \beta_i \geq 0$. Hence, we can calculate (13) as

$$v_i = \{a_i + A_i(\alpha_i p_i + \beta_i p_{i+1})\} \cdot \frac{p_i - p_{i+1}}{||p_i - p_{i+1}||}$$

$$= -\omega_i\sqrt{(p_i^1 - p_{i+1}^1)^2 + (p_i^2 - p_{i+1}^2)^2}. \qquad (14)$$

Note that λ_i does not appear in (14). From (14), we can see that the parameters α_i and β_i vanish, and hence v_i is constant at any point $x \in D_i$. Since v_i does not vanish at any point $x \in D_i$, the system (2) with (28) does not have any equilibrium points in $M(\varepsilon)$. $\square$

Now, a definition on the concept "traversal" for the system (2) with (28) is given as follows [11].

Definition 2 : Let Σ be a line segment in the positively invariant, bounded and closed set $M(\varepsilon)$. If the value of an inner product of the unit normal vector to Σ: e_Σ and the velocity vector of the N-modal and 2-dimensional piecewise affine system (2) with (28) is not equal to 0 and its sign does not change at any point in Σ, then Σ is said to be *traversal* with respect to the system (2) with (28). $\square$

In addition to Lemma 2, under the condition of $\omega_i > 0(i = 1, \cdots, N)$, a solution trajectory vector of the system (2) with (28) always has a velocity component in the counterclockwise

rotation. On the other hand, under the condition of $\omega_i < 0 (i = 1, \cdots, N)$, a solution trajectory vector of (2) with (28) always has a velocity component in the clockwise rotation. From this fact, we can derive the following lemma.

Lemma 3 : For the N-modal and 2-dimensional piecewise affine system (2) with (28), assume that $\omega_i > 0 (i = 1, \cdots, N)$ or $\omega_i < 0$ $(i = 1, \cdots, N)$ holds. Then, there exists a traversal line segment Σ at any point in $x \in M(\varepsilon)$, and it is satisfied that $x \in \Sigma$ and Σ infinitely intersects with solution trajectories of the system (2) with (28).

(Proof) We assume that $\omega_i > 0$ $(i = 1, \cdots, N)$ or $\omega_i < 0$ $(i = 1, \cdots, N)$ holds. Then, a solution trajectory of the system (2) with (28) always circles to the counterclockwise rotation or to the clockwise rotation. Now, for a point $x \in M$, we consider a half line whose origin is O and that passes through x, and define a subset $\Sigma \subset M$ as the intersection of the half line and M. Since the velocity vector field of the system (2) with (28) always has the velocity component of the counterclockwise rotation or the clockwise rotation, the inner product of a normal vector of Σ and the vector field of the system (2) with (28) at any point in Σ is not equal to 0 and its sign does not change, that is, Σ is traversal. Moreover, in each M_i $(i = 1, \cdots, N)$, since the velocity vector field of the system (2) with (28) always has the velocity component of the clockwise rotation or the counterclockwise rotation, a solution trajectory of the system (2) with (28) $x(t)$ that intersected Σ intersects Σ in a finite time again. Consequently, it turns out the solution trajectory intersects Σ infinitely. □

We have to note that the result in Lemma 3 does not depend on λ_i $(i = 1, \cdots, N)$. Using Lemmas 1–3, we can derive the main theorem on the existence of the limit cycle of the system (2) with (28).

Theorem 2 : For the N-modal and 2-dimensional piecewise affine system (2) with (28), assume that $\omega_i > 0 (i = 1, \cdots, N)$ or $\omega_i < 0$ $(i = 1, \cdots, N)$ holds. Then, the unique and stable limit cycle of the system (2) with (28) is equivalent to C.

(Proof) By the result on the hybrid Poincare-Bendixson theorem derived in [3], [11], it turns out that sufficient conditions for the existence of stable limit cycles of the system (2) with (28) in $M(\varepsilon)$ are the following three: (i) $M(\varepsilon)$ is a positively invariant, bounded and closed set, (ii) there do not exist any equilibrium points at the boundary and in the interior of $M(\varepsilon)$ (iii) there exists a traversal line segment $\Sigma \subset M(\varepsilon)$ such that $x \in \Sigma$ and Σ infinitely intersects with solution trajectories of the system (2) with (28). Since we have confirmed these three conditions in Lemma 1, 2, and 3, we can see that there exists a stable limit cycle in $M(\varepsilon)$ for the system (2) with (28) for any admissible ε. Moreover, since $M(\varepsilon)$ converges to C as the values of ε goes to 0, it can be confirmed that C is a unique and stable limit cycle. Hence, the proof is completed. □

In this paper, we consider an additional parameter λ_i in the system (2), (28). However, from the results obtained in this subsection, we can see that the existence and the uniqueness of a limit cycle of the system (2), (28) are independent of λ_i. This fact is quite important in the next section.

C. Theoretical Analysis

Finally, this subsection gives theoretical analysis on rotational directions and periods of limit cycle solution trajectories of the system (2) with (28). First, we consider the relationship between rotational directions of limit cycles and the parameters in (28). The following proposition can be derived.

Proposition 1 : For the N-modal and 2-dimensional piecewise affine system (2) with (28), its limit cycle solution trajectory moves in the counterclockwise rotation for $\omega_i > 0$ $(i = 1, \cdots, N)$, and conversely it moves in the clockwise rotation for $\omega_i < 0$ $(i = 1, \cdots, N)$.

(Proof) The proof of this proposition is trivial from the discussion in the previous section. □

From Proposition 1, it is confirmed that the rotational directions of limit cycles do not depend on λ_i. Next, we analyze periods of limit cycles of the system (2) with (28). It can be expected that after a solution trajectory of the system (2) with (28) converges to C, it behaves as a periodic trajectory. By calculating the velocity component of the vector of the system along C, we can derive the next proposition.

Proposition 2 : When a limit cycle solution trajectory of the N-modal and 2-dimensional piecewise affine system (2) with (28) is sufficiently close to C, the period with which it rotates around C is given by

$$T \approx \sum_{i=1}^{N} \frac{1}{|\omega_i|}. \tag{15}$$

(Proof) The velocity component of a solution trajectory v_i of (2) with (28) in D_i to the direction of $p_i - p_{i+1}$ is given by (13). The length of C_i: L_i can be calculated as

$$L_i = \sqrt{(p_i^1 - p_{i+1}^1)^2 + (p_i^2 - p_{i+1}^2)^2}. \tag{16}$$

Therefore, we can obtain the period T as

$$T \approx \sum_{i=1}^{N} \frac{L_i}{|v_i|} = \sum_{i=1}^{N} \frac{1}{|\omega_i|}. \tag{17}$$

This completes the proof of this proposition. □

From Proposition 2, we can also see that the period of a limit cycle solution trajectory of the system (2), (28) is not independent of λ_i. So, we can freely choose the value of λ_i.

III. LIMIT CYCLE CONTROL FOR PIECEWISE AFFINE SYSTEMS

A. Formulation of Limit Cycle Control

In this section, we consider a controller design problem on generation of limit cycles for given piecewise affine control systems. First, this sub-section gives the problem formulation. Consider the next piecewise affine control system defined in D_i:

$$\dot{x} = a_i + A_i x + b_i u, \ \ x \in D_i, \tag{18}$$

where $u \in \mathbf{R}$ is the control input and $b_i \in \mathbf{R}^2$ is the coefficient vector for the control input. We next consider the state feedback law:

$$u = k_i x + l_i, \ \ x \in D_i, \tag{19}$$

where $k_i \in \mathbf{R}^2$ and $l_i \in \mathbf{R}$. We assume that p_i, p_{i+1}, a_i, A_i are given parameters. Now, we formulate a problem on generating a desired limit cycle for the piecewise affine control system (2) and the state feedback law (19) as follows.

Problem 2 : For the N-modal and 2-dimensional piecewise affine control system (18) with the state feedback law (19), design $b_i, \ k_i, \ l_i, \omega_i, \lambda_i \ (i = 1, \cdots, N)$ such that a given polygonal closed curve C (1) is a unique and stable limit cycle of the closed-loop system. □

Throughout this paper, we call Problem 2 *a limit cycle control problem for piecewise affine control system.*

B. Matching Conditions for Limit Cycle Control Problem

The purpose of this subsection is to derive a solution method of Problem 2 for the piecewise affine control system (18) with the state feedback law (19). To fulfill this, we shall utilize the limit cycle synthesis method obtained in Section 2. The results in Section II show that the unique and stable limit cycle of the system (2), (28) coincides with C. Hence, by tuning design parameters $b_i, \ k_i, \ l_i, \omega_i, \lambda_i \ (i = 1, \cdots, N)$, we conform the closed-loop system (18), (19) to the system (2), (28). We here call the system (2), (28) *the reference system.* Use the following notations for the system (2), (28):

$$a_i = \begin{bmatrix} a_i^1 \\ a_i^2 \end{bmatrix}, \ \ A_i = \begin{bmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{bmatrix},$$
$$b_i = \begin{bmatrix} b_i^1 \\ b_i^2 \end{bmatrix}, \ \ k_i = \begin{bmatrix} k_i^1 & k_i^2 \end{bmatrix}. \tag{20}$$

Conditions such that the closed-loop system (18), (19) is consistent with the reference system (2), (28) can be obtained by the following theorem.

Theorem 3 : The N-modal and 2-dimensional piecewise affine control system (18) with the state feedback control law (19) is equivalent to the reference system (2), (28) if and only if *the matching conditions*:

$$a_i^1 + b_i^1 l_i = -\lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1)$$
$$-\omega_i(p_i^1 - p_{i+1}^1) \quad (21)$$
$$a_i^2 + b_i^2 l_i = \lambda_i(p_i^1 - p_{i+1}^1)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1)$$
$$-\omega_i(p_i^2 - p_{i+1}^2) \quad (22)$$
$$A_i^{11} + b_i^1 k_i^1 = -\lambda_i(p_i^2 - p_{i+1}^2)^2 \tag{23}$$
$$A_i^{12} + b_i^1 k_i^2 = \lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) \tag{24}$$
$$A_i^{21} + b_i^2 k_i^1 = \lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) \tag{25}$$
$$A_i^{22} + b_i^2 k_i^2 = -\lambda_i(p_i^1 - p_{i+1}^1)^2 \tag{26}$$

hold.

(Proof) Substituting (19) into (18), we get the closed-loop system:

$$\dot{x} = a_i + A_i x + b_i(k_i x + l_i)$$
$$= a_i + b_i l_i + (A_i + b_i k_i)x$$
$$= \begin{bmatrix} a_i^1 + b_i^1 l_i \\ a_i^2 + b_i^2 l_i \end{bmatrix} + \begin{bmatrix} A_i^{11} + b_i^1 k_i^1 & A_i^{12} + b_i^1 k_i^2 \\ A_i^{21} + b_i^2 k_i^1 & A_i^{22} + b_i^2 k_i^2 \end{bmatrix} x. \tag{27}$$

Comparing the components of the reference system (2), (28) to the closed-loop system (27), we can obtain the matching conditions (21)–(26). □

The matching conditions (21)–(26) consists of 6 algebraic equations, and 7 unknown variables: $b_i^1, b_i^2, k_i^1, k_i^2, l_i, \omega_i, \lambda_i$. Hence, by solving them under the condition $\lambda_i > 0$, we can obtain these unknown variables, that is, a solution of Problem 2.

IV.   SIMULATIONS

This section presents a numerical example in order to confirm the effectiveness of the results derived in the previous sections. We now give data of a polygon with $N = 4$ as $P_1 = (1, 0), \ P_2 = (0, 1), \ P_3 = (-1, 0), \ P_4 = (0, -1).0$ The polygon is shown in Fig. 4.
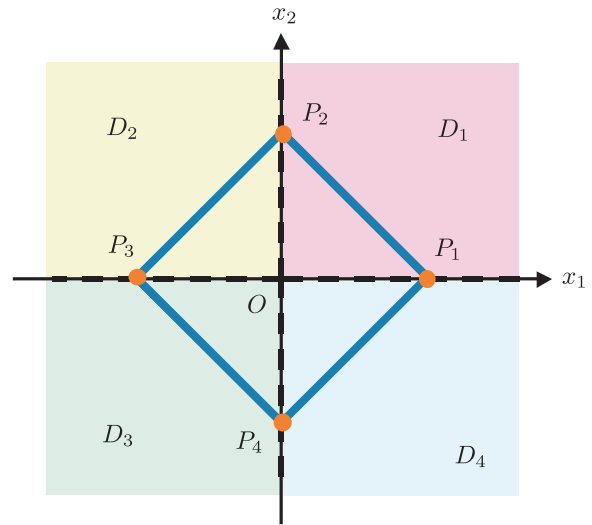

Fig. 4 : Polygonal Closed Curve of Example

The coefficients of the piecewise affine system are given by

$$a_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix},$$
$$a_2 = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},$$
$$a_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ A_3 = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix},$$
$$a_4 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}, \ A_4 = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}. \tag{28}$$

By solving the matching conditions for Problem 2 (21)–(26) in terms of the problem formulation, we can calculate

design parameters as follows:

$$b_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ k_1 = \begin{bmatrix} -1 & -2 \end{bmatrix},$$
$$l_1 = 1, \ \omega_1 = 3, \ \lambda_1 = 1,$$
$$b_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \ k_2 = \begin{bmatrix} 1 & -2 \end{bmatrix},$$
$$l_2 = 2, \ \omega_2 = 1, \ \lambda_2 = 2,$$
$$b_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \ k_3 = \begin{bmatrix} 1 & -2 \end{bmatrix}, \tag{29}$$
$$l_3 = 2, \ \omega_3 = 2, \ \lambda_3 = 3,$$
$$b_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ k_4 = \begin{bmatrix} 4 & 4 \end{bmatrix},$$
$$l_4 = -4, \ \omega_4 = 1, \ \lambda_4 = 4.$$

Note that $\lambda_i > 0$ $(i = 1, 2, 3, 4)$ holds in (29). It can be confirmed that from Proposition 1, a limit cycle solution trajectory moves in the counterclockwise rotation since $\omega_i > 0$ $(i = 1, 2, 3, 4)$ holds. In addition, from Proposition 2 we can estimate the period of a limit cycle solution trajectory as

$$T \approx \sum_{i=1}^{4} \frac{1}{|\omega_i|} = \frac{17}{6}. \tag{30}$$

We set the initial state as $x_0 = \begin{bmatrix} 1, & 1 \end{bmatrix}^\mathsf{T}$ for the numerical simulation. The simulation results are illustrated in Figs. 5–7. Fig. 5 shows the solution trajectory on the x_1x_2-plane. In Figs. 6 and 7, the time series of x_1 and x_2 are shown, respectively. From these simulation results, we can see that the solution trajectory that starts from x_0 behaves as a limit cycle for the desired polygonal closed curve C, and hence Theorem 1 holds. As we expected above, the solution trajectory moves in the counterclockwise rotation, and this result is coincident with Proposition 1. Moreover, the estimated period $T \approx 17/6$ is mostly agree about the simulation result from Figs. 6 and 7.
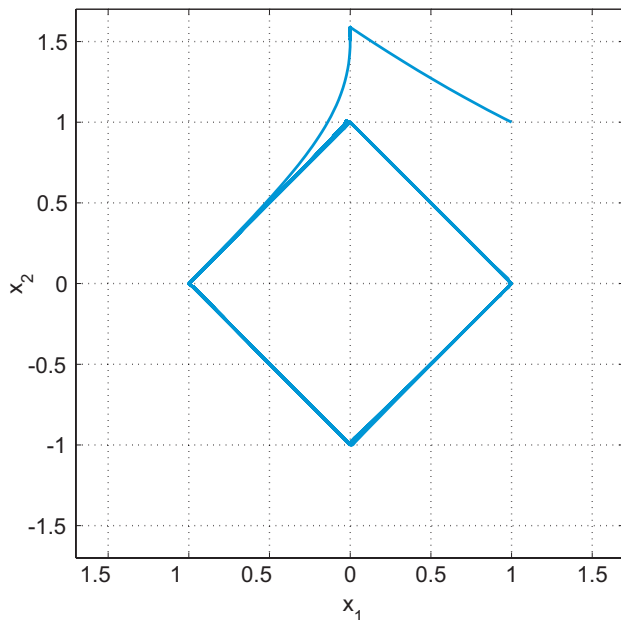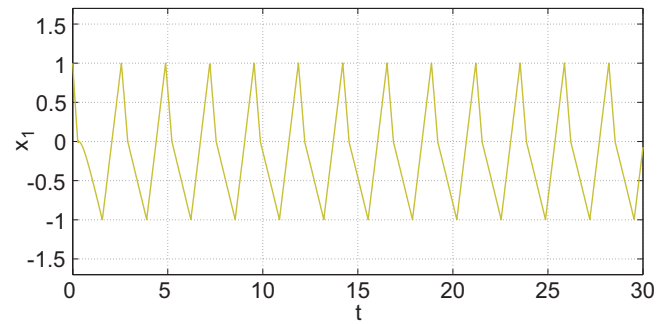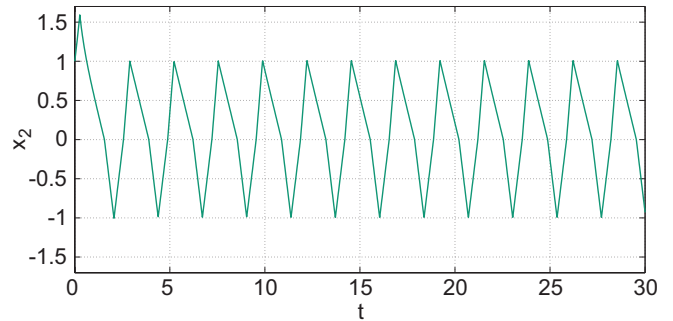

Fig. 6 : Time Series of x_1

Fig. 7 : Time Series of x_2

V. CONCLUSION

In this paper, we have considered a limit cycle control problem for a multi-modal and 2-dimensional piecewise control affine system. We have derive the matching conditions such that the piecewise control affine system with the state feedback law corresponds with the reference system which generates a unique and stable limit cycle. It has been confirmed by solving the matching conditions we can obtain the values of design parameters. A numerical simulations show the availability and the application potentiality of the proposed method.

Our future work includes applications of the proposed control method to real systems and extensions to multi-dimensional piecewise affine systems.

Fig. 5 : Solution Trajectory on x_1x_2-Plane

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